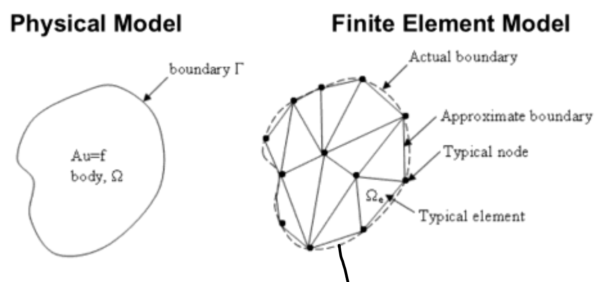
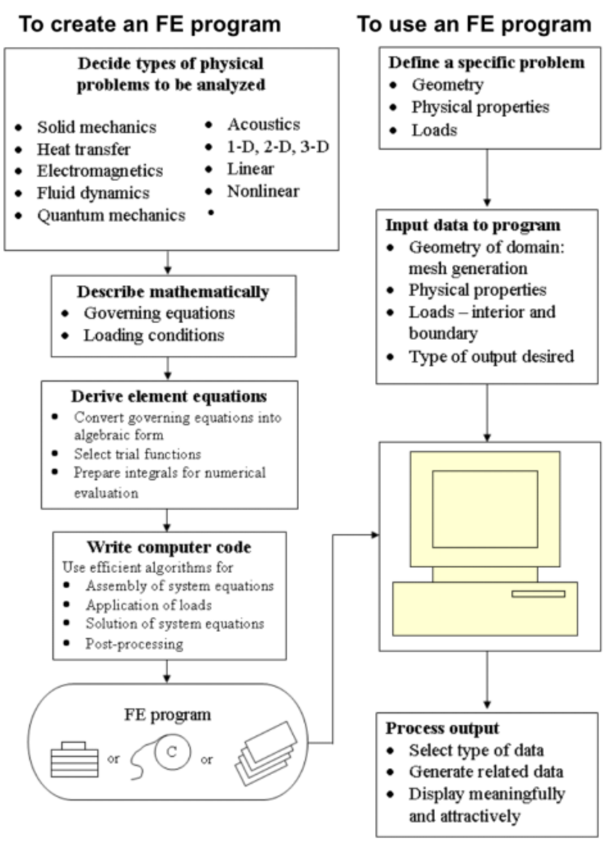
Intro

Tuesday, August 20, 2024 11:37 AM



Discretization error

Ω - domain



Discrete systems are characterized by a set of algebraic equations

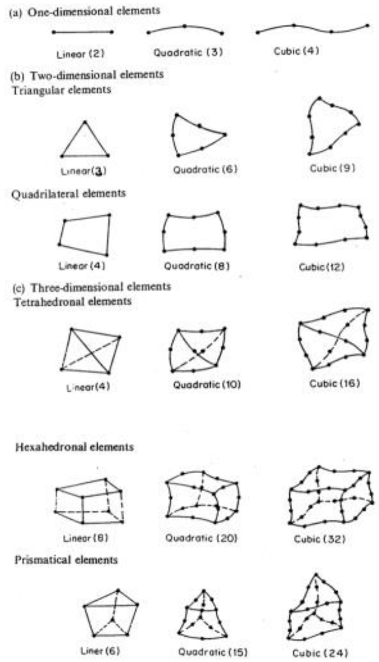
Continuous systems are characterized by a set of partial differential equations

Free vibrations ⇒ eigenvalue problem ⇒ non-trivial solution

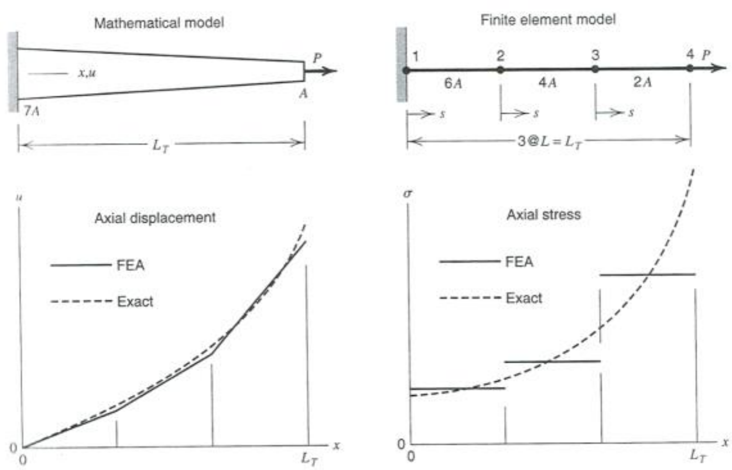
↑load does not stay with system

where in transient response it does

SHAPES OF SOME CLASSICAL ELEMENTS



TAPERED BAR DISCRETIZED BY THREE UNIFORM TWO-NODE ELEMENTS



$u \rightarrow$ displacement

$\frac{\partial u}{\partial x} \rightarrow$ strain

$E \frac{\partial u}{\partial x} \rightarrow$ stress

$\uparrow$ Young's modulus

accuracy increases as # of elements or # of nodes increases

How is g-code read?

Multi scale - important features at multiple scales of time and/or space

Multi physics - multiple physical models or multiple simultaneous physical phenomena

Classical Variational Methods

Tuesday, August 27, 2024 11:24 AM

Matrix approach works only for structural analysis

Governing equation for a beam

- Equilibrium method ⇒ FBD
- Variational method ⇒ General

Differential equations and boundary conditions

C^0 Variational Problem

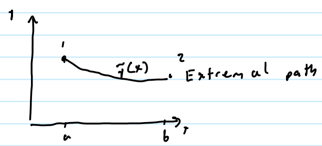
δ - variational operator

ϵ - arbitrary parameter independent of x

I - functional: function of functions that give a specific value for a given function $F(x)$

Ex. $I = \int_0^5 F\,dx \Rightarrow$ for $F = x^2$, $I = 41.67$

$n(a) = n(b) = 0$, fixed point problem



$$I = \int_a^b F(x, y, y')\,dx$$

$$I = I(\epsilon) = \int_a^b F(x, \tilde{y} + \epsilon y, \tilde{y}' + \epsilon y')\,dx$$

$$\frac{dI}{d\epsilon} = \frac{d}{d\epsilon}\int_a^b F(x, \tilde{y} + \epsilon y, \tilde{y}' + \epsilon y')\,dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial(\tilde{y} + \epsilon y)} \frac{\partial(\tilde{y} + \epsilon y)}{\partial \epsilon} + \frac{\partial F}{\partial(\tilde{y}' + \epsilon y')} \frac{\partial(\tilde{y}' + \epsilon y')}{\partial \epsilon} \right) d$$

$$\frac{\partial F}{\partial x} \cancel{\frac{\partial x}{\partial \epsilon}}^{\,0}$$

ϵ does not depend on x

$$= \int_a^b \left[\frac{\partial F}{\partial(\tilde{y} + \epsilon y)} n \quad \frac{\partial F}{\partial(\tilde{y}' + \epsilon n')} n' \right] dx$$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left[n \frac{\partial F}{\partial y} + n' \frac{\partial F}{\partial y'} \right] dx$$

$y = \tilde{y} + \epsilon n$
at $\epsilon = 0$, $y = \tilde{y}$

$$\delta I = \epsilon \left[\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} \right]$$

$$\delta I = \int_a^b \left[\epsilon n \frac{\partial F}{\partial y} + \epsilon y' \frac{\partial F}{\partial y'} \right] dx$$

$$\delta I = 0 = \int_a^b \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx$$

$\delta y(a) = \delta y(b) = 0$

Integrate second term by parts once

$$\int_a^b \frac{\partial F}{\partial y'} \frac{d}{dx}(\delta y)\,dx = \frac{\partial F}{\partial y'} \cancel{\delta y}\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y\,dx$$

0 since $\delta y(a) = \delta y(b) = 0$, fixed points

$$\int_a^b v \frac{du}{dx}\,dx = vu\Big|_a^b - \int_a^b \frac{dv}{dx} u\,dx \qquad \delta y' = \delta \frac{dy}{dx} = \frac{d}{dx}(\delta y)$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{Euler - Lagrange Eqn}$$

C^1 variational problem:

Highest derivative is of order 2

$$I = \int_a^b F(x, y, y', y'')\,dx$$

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y''\right)dx = 0$$

$$\delta y' = \frac{d}{dx}(\delta y)$$

$$\delta y'' = \frac{d^2}{dx^2}(\delta y)$$

Integrate 2nd and 3rd term by parts (once, twice respectively)

$$\delta I = \int_a^b \frac{\partial F}{\partial y}\delta y + \left[\frac{\partial F}{\partial y'}\delta y\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx\right] + \underbrace{\left[\frac{\partial F}{\partial y''}\delta y'\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\,dx\right]}_{\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y\Big|_a^b - \int_a^b \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\,dx\right]} = 0$$

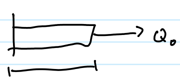
$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y\,dx + \frac{\partial F}{\partial y'}\delta y\Big|_a^b + \frac{\partial F}{\partial y'}\delta y\Big|_a^b + \frac{\partial F}{\partial y''}\delta y'\Big|_a^b + \frac{\partial F}{\partial y''}\delta y'\Big|_a^b - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y\Big|_a^b = 0$$

$$\delta I = \int_a^b \underbrace{\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right]}_{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0}\delta y\,dx + \left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y\Big|_a^b + \frac{\partial F}{\partial y''}\delta y'\Big|_a^b = 0$$

Essential boundary condition		Natural Boundary conditions
δy	or	$\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)$
$\delta y'$	or	$\frac{\partial F}{\partial y''}$

Example Bar:

$EA\,\frac{d^2u}{dx^2} = q(x) \qquad 0 < x < L$



E - Young's modulus

A - cross sectional area

BC: $\sigma = E\varepsilon = E\,\frac{du}{dx}$

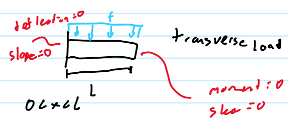
EBC $\rightarrow$ $x = 0,\ u(0) = 0$ homogeneous BC

NBC $\rightarrow$ $x = L \quad \frac{du}{dx}(L) = Q_0$

$Q = \sigma A$

u: axial displacement

C_1 Example Beam:



$EI\,\frac{d^4w}{dx^4} = f \qquad 0 < x < L$

BC:

@ $x = 0$ @ $x = L$ w: transverse displacement

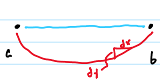
deflection $w(0) = 0$ $\frac{d^3w}{dx^3}(L) = 0$ shear

slope $\frac{dw}{dx}(0) = 0$ $\frac{d^2w}{dx^2}(L) = 0$ moment $V = \frac{dM}{dx}$

Example problem 1:

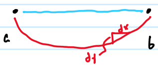
Find shortest distance between two points by considering extremum

$$I = S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$



$ds^2 = dx^2 + dy^2$

$$I = S = \int_a \sqrt{1+\left(\frac{dy}{dx}\right)}\, dx$$



$$ds^2 = dx^2 + dy^2$$

$$ds^2 = 1+\left(\frac{dy^2}{dx^2}\right)^2 dx^2$$

$$ds = \sqrt{1+\left(\frac{dy^2}{dx^2}\right)^2}\, dx$$

$$\delta I = \delta S = \int_a^b \left(\frac{dF}{dy}\delta y + \frac{dF}{dy'}\delta y' \right) dx = 0$$

Weak or Variational Formulation

$$-\frac{d^2u}{dx^2} + u = q(x) \qquad 0 < x < 1$$

B.C. $\quad u(0) = u(1) = 0$

Ex. Given $-\frac{d}{dx}\left(a\frac{du}{dx}\right) - cu + x^2 = 0$

B.C. $\quad u(0) = 0, \left(a\frac{du}{dx}\right)\Big|_{x=1} = 1$

Find weak form

multiply by weight function v and integrate between (0,1)

$$0 = \int_0^1 v\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) - cu + x^2\right] dx$$

$$0 = \int_0^1 \left(a\frac{dv}{dx}\frac{du}{dx} - cuv + vx^2\right) dx - \left(va\frac{du}{dx}\right)\Big|_0^1$$

$$\int_0^1 v\frac{d}{dx}\left(a\frac{du}{dx}\right) dx = v\left(a\frac{du}{dx}\right)\Big|_0^1 - \int_0^1 \frac{dv}{dx}\, a\frac{du}{dx}$$

$V = \delta u$

$\Rightarrow V(0) = 0$

B.C.'s $\quad u = 0$ at $x = 0$

$a\frac{du}{dx} = 1$ at $x = 1$

$$0 = \int_0^1 \left(a\frac{dv}{dx}\frac{du}{dx} - cvu\right) dx + \int_0^1 vx^2\, dx - v(1)$$

This lowers the requirement from the eq. eqn to be differentiable twice to once in the weak form.

$$0 = B(v,u) - \ell(v)$$

$B(v,u) = \int_0^1 \left(a\frac{dv}{dx}\frac{du}{dx} - cvu\right) dx \Rightarrow$ bilinear

$\ell(v) = \int_0^1 vx^2\, dx + v(1) \Rightarrow$ linear

If $B(v,u) = B(u,v)$, called symmetric bilinear

If $B(\cdot,\cdot)$ is symmetric and bilinear, and $\ell(v)$ is linear

there is a quadratic functional with weak form of

$$I(u) = \frac{B(u,v)}{2} - \ell(u)$$

$$I(u) = \frac{1}{2}\int_0^1 \left[a\left(\frac{du}{dx}\right)^2 - cu^2\right] dx + \int_0^1 ux^2\, dx - u(1)$$

$$I(u) = \frac{1}{2}\int_0^1 \left[a\left(\frac{du}{dx}\right)^2 - cu^2 + 2ux^2\right] dx - u(1)$$

looks like total potential energy principle

Classical Variational Methods

Ritz method: consider weak form to approximate

Ex.

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0$$

$u(0) = 0;\ u(1) = 1$

weak form:

$$0 = \int_0^1 v \left\{ -\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] \right\} dx$$

$v = \delta u$

$v(0) = v(1) = 0$

$$= \int_0^1 (1+x) \frac{dv}{dx} \frac{du}{dx} \, dx - \left[\cancel{v}^{\,0} (1+x) \frac{du}{dx} \right]_0^1$$

$$= \int_0^1 (1+x) \frac{dv}{dx} \frac{du}{dx} \, dx \qquad \text{weak form}$$

let $u = \sum_{j=1}^{N} c_j \phi_j + \phi_0$

$\phi_0(0) = 0$, $\phi_0(1) = 1$

choose $\phi_0 = x$

satisfies non-homogenous EBC

$\phi_j(0) = \phi_j(1) = 0$

$\phi_1 = x(1-x)$

$\phi_2 = x^2(1-x)$

$\phi_j = x^j(1-x)$

satisfies homogenous EBC

$$0 = \int_0^1 (1+x) \frac{d\phi_i}{dx} \left[\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right] dx$$

$$u = \sum_{j=1}^{N} \phi_j c_j + \phi_0$$

$$\frac{du}{dx} = \sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx}$$

$$v = \delta u = \delta\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0 \right)$$

$$v = \frac{\partial}{\partial c}\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0 \right) = \delta_{ij} \phi_j = \phi_i$$

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replace $\phi_0 = x$, $\phi_j = x^j(1-x)$

$$0 = \sum_{j=1}^{N} \left\{ \int_0^1 (1+x) \underbrace{\left[i x^{i-1} - (i+1) x^i \right]}_{\frac{d\phi_i}{dx}} \underbrace{\left[j x^{j-1} - (j+1) x^j \right]}_{\frac{d\phi_j}{dx}} dx \; c_j + \int_0^1 (1+x) \underbrace{\left[i x^{i-1} - (i+1) x^i \right]}_{\frac{d\phi_i}{dx}} \underbrace{(1)}_{\frac{d\phi_0}{dx}} dx \right.$$

$$0 = \sum_{j=1}^{N} B_{ij} c_j - F_i$$

where $B_{ij} = \int_0^1 (1+x) \left[i x^{i-1} - (i+1) x^i \right] \left[j x^{j-1} - (j+1) x^j \right] dx$

Expand

Simplify

$$B_{ij} = \frac{ij}{i+j-1} - \frac{ij + i + j}{i+j} + \frac{1 - ij}{i+j+1} + \frac{(i+1)(j+1)}{i+j+2}$$

$$F_i = \frac{1}{(1+i)(2+i)}$$

$N = 2$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$B_{11} = 1 - \frac{3}{2} + 0 + \frac{4}{4} = \frac{1}{2}$$

$$B_{12} = \frac{2}{2} - \frac{5}{3} - \frac{1}{4} + \frac{6}{5} = \frac{17}{6}$$

$$B_{21} = B_{12} = \frac{17}{6}$$

$$B_{22} = \frac{4}{3} - 2 - \frac{3}{2} + \frac{9}{1} = \frac{7}{30}$$

$$F_1 = \frac{1}{6}$$

$$F_2 = \frac{1}{12}$$

$$\frac{1}{60} \begin{bmatrix} 30 & 17 \\ 17 & 14 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{12} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}$$

$$c_1 = \frac{55}{131} \qquad c_2 = -\frac{20}{131}$$

Ritz solution:

$$u = \sum_{j=1}^{N} c_j \phi_j + \phi_0$$
$$= c_1\phi_1 + c_2\phi_2 + \phi_0$$
$$= \frac{55}{131}(x-x^2) - \frac{20}{131}(x^2-x^3) + x$$
$$= \frac{1}{131}(186x - 75x^2 + 20x^3)$$

Exact solution (given):

$$u_{exact} = \frac{\log_{10}(1+x)}{\log_{10} 2}$$

for $x = \frac{1}{2}$, $u_{Ritz} = 0.5858778$

$u_{exact} = 0.5844625$

Obtain one parameter solution using quadratic functional:

Weak form:

$$0 = \int_0^1 (1+x)\frac{dv}{dx}\frac{du}{dx}\,dx$$
$$= B(v,u) - \cancel{l(v)}^{0}$$

$B(u,v) = B(v,u)$ ✓ Bilinear and symmetric

Quadratic functional:

$$I(u) = \frac{1}{2}B(u,v) - \cancel{l(u)}^{0}$$
$$I = \frac{1}{2}\int_0^1 (1+x)\left(\frac{du}{dx}\right)^2 dx$$

$u = c_1\phi_1 + \phi_0$, $\phi_0 = x$, $\phi_1 = x(1-x)$

$u = c_1(x-x^2) + x$

$$\frac{du}{dx} = c_1(1-2x) + 1$$
$$= 1 + c_1 - 2c_1 x$$

$$I = \frac{1}{2}\int_0^1 (1+x)(1 + 2c_1 - 4c_1x + c_1^2 - 4c_1^2 + 4c_1^2x^2)\,dx$$

$$I = I(c_1) = \frac{1}{2}\left(\frac{3}{2} - \frac{c_1}{3} + \frac{c_1^2}{2}\right)$$

Extremum value $\frac{dI}{dc_1} = \frac{1}{2}\left(-\frac{1}{3} + \frac{2}{2}c_1\right) = 0$

$$\Rightarrow c_1 = \frac{1}{3}$$
$$u = c_1\phi_1 + \phi_0$$
$$u = \frac{1}{3}(x-x^2) + x$$

$N=1$, $B_{11}c_1 = F_1$

$\frac{1}{2}c_1 = \frac{1}{6}$

$c_1 = \frac{1}{3}$

Weighted residual methods

Ω : domain

S : boundary

Direct solving, no weak formula

Residual / Error function:

$$R = A\left(\sum_{j=1}^{N} c_j\phi_j + \phi_0\right) - f \neq 0$$

Make R as small as possible

ψ : weight function

3 methods:

Galerkin:

set $\psi_i = \Phi_i \Rightarrow \int_\Omega \Phi_i R\, dx\, dy = 0$

weight function = approximation function

Least squares:

$\delta \int_\Omega R^2\, dx\, dy = 0 \Rightarrow \int_\Omega R \frac{\partial R}{\partial c_i}\, dx\, dy = 0$

Collocation:

$R = 0$ at selected points (x_i, y_i)

Ex.

$\frac{d^4 u}{dx^4} + u - 1 = 0$

- C_1 variation 1
- order 4
- functional order 2
- $2m = 4$
- $m = 2 > 2 - 1 \Rightarrow NBC$

BC: $u(0) = u(1) = 0$

$u''(0) = u''(1) = 0$

Determine one parameter solution using.

a. Galerkin method:

$u_1 = \Phi_0 + c_1 \Phi_1$

$\int_\Omega \psi_i R\, dx\, dy = 0$

$\psi_i = \Phi_i$, Galerkin

$\Phi_0 = 0$

From BC

$\Phi_1 = \sin \pi x$

$u = c_1 \sin \pi x$

$\frac{du}{dx} = \pi c_1 \cos \pi x$

$\frac{d^2 u}{dx^2} = -\pi^2 c_1 \sin \pi x$

$\cdots = -\pi^3 c_1 \cos \pi x$

$\cdots = \pi^4 c_1 \sin \pi x$

$R = \frac{d^4 u_1}{dx^4} + u_1 - 1$

$= c_1 \pi^4 \sin \pi x + c_1 \sin \pi x - 1$

for $\psi_i = \Phi_i$

$\int_0^1 \Phi_1 R(c_1)\, dx = 0$

$= \int_0^1 \sin \pi x \,(c_1 \pi^4 \sin \pi x + c_1 \sin \pi x - 1)\, dx = 0$

$c_1 = \frac{4}{\pi}\left(\frac{1}{1+\pi^4}\right)$

$u_1 = \cancel{\Phi_0}^{\,0} + c_1 \Phi_1$

$\boxed{u_1 = \frac{4}{\pi}\left(\frac{1}{1+\pi^4}\right) \sin \pi x}$

$u_{exact} = 1 - \cos \frac{x}{\sqrt{2}} \cosh \frac{x}{\sqrt{2}} + 0.38 \cos \frac{x}{\sqrt{2}} \sinh \frac{x}{\sqrt{2}} - .322 \sin \frac{x}{\sqrt{2}} \cosh \frac{x}{\sqrt{2}}$

b. Least squares:

$\psi_i = \frac{\partial R}{\partial c_i}$

$\int_0^1 \frac{\partial R}{\partial c_1} R\, dx = 0$

$R = c_1 \pi^4 \sin \pi x + c_1 \sin \pi x - 1$

$u = \cancel{\Phi_0}^{\,0} + c_1 \Phi_1$

$\frac{\partial R}{\partial c_1} = \pi^4 \sin \pi x + \sin \pi x$

$\int_0^1 (\pi^4 \sin \pi x + \sin \pi x)(c_1 \pi^4 \sin \pi x + c_1 \sin \pi x - 1)\, dx$

$c_1 = \frac{4}{\pi}\left(\frac{1}{1+\pi^4}\right)$

$\boxed{u = \frac{4}{\pi}\left(\frac{1}{1+\pi^4}\right) \sin \pi x}$

c. Collocation method:

$x=0$ —— $x = \frac{1}{2}$ —— $x=1$

$R\left(\frac{1}{2}\right) = 0$

$c_1 \pi^4 \sin \frac{\pi}{2} - c_1 \sin \frac{\pi}{2} - 1 = 0$

$c_1 = \frac{1}{1+\pi^4} \sin \pi x$

$\boxed{u_1 = \frac{1}{1+\pi^4} \sin \pi x}$

Homework 1

Thursday, September 5, 2024 11:03 AM

C-Homewor
k+1

Due: September 12,2024 NAME: Easton Ingram

HOMEWORK SET # 1
ME/AE 5212 Introduction to Finite Element Analysis

Construct the weak form and, whenever possible, quadratic functional.

1.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) = f \qquad \text{for } 0 < x < L$$

$$u(0) = 0, \qquad \left(a\frac{du}{dx} + ku\right)\Big|_{x=L} = P$$

(10 points)

2. (# 2.5 Text)

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \quad \text{for } 0 < x < 1$$

$$\left(u\frac{du}{dx}\right)\Big|_{x=0} = 0, \qquad u(1) = \sqrt{2}$$

(10 points)

1.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f=0$$

$$0=\int_0^L v\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f\right]dx$$

$$=\int_0^L\left[-v\frac{d}{dx}\left(a\frac{du}{dx}\right)-fv\right]dx$$

$$=\int_0^L\left(a\frac{du}{dx}\frac{dv}{dx}-fv\right)dx-va\frac{du}{dx}\Big|_0^L$$

$$va\frac{du}{dx}\Big|_0^L=v(L)\left(P-ku|_{x=L}\right)$$

$$\left(a\frac{du}{dx}+ku\right)\Big|_{x=L}=P \Rightarrow a\frac{du}{dx}\Big|_{x=L}=P-ku|_{x=L}$$

$$\int w\,dz=wz-\int z\,dw$$

$$w=v$$

$$z=\frac{d}{dx}\left(a\frac{du}{dx}\right)$$

$$\int -v\frac{d}{dx}\left(a\frac{du}{dx}\right)dx$$

$$=-va\frac{du}{dx}\Big|_0^L-\int a\frac{du}{dx}\left(-\frac{dv}{dx}\right)$$

$$=\int_0^L\left(a\frac{du}{dx}\frac{dv}{dx}\right)-\left(va\frac{du}{dx}\right)\Big|_0^L$$

$$0=\int_0^L\left(a\frac{du}{dx}\frac{dv}{dx}-fv\right)dx-v(L)\left(P-ka|_{x=L}\right)$$

$$\boxed{0=\int_0^L\left(a\frac{du}{dx}\frac{dv}{dx}\right)dx-\int_0^L(fv)\,dx-v(L)\left(P-ka|_{x=L}\right)}$$

quadratic functional:

$$0=B(v,u)-\ell(v)$$

$$B(v,u)=\int_0^L\left(a\frac{du}{dx}\frac{dv}{dx}\right)dx$$

$$\ell(v)=\int_0^L(fv)\,dx+v(L)\left(P-ka|_{x=L}\right)$$

$$I(u)=\frac{B(u,u)}{2}-\ell(u)$$

$$I(u)=\frac{1}{2}\int_0^L a\left(\frac{du}{dx}\right)^2dx-\int_0^L(fu)\,dx-u(L)\left(P-ka|_{x=L}\right)$$

$$\boxed{I(u)=\frac{1}{2}\int_0^L\left[a\left(\frac{du}{dx}\right)^2-\frac{1}{2}fu\right]dx-u(L)\left(P-ka|_{x=L}\right)}$$

2.

$$0 = \int_0^1 v\left[-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f\right]dx$$

$$= \int_0^1 \left(-v\frac{d}{dx}\left(u\frac{du}{dx}\right)+fv\right)dx$$

$$= \int_0^1 \left(u\frac{du}{dx}\frac{dv}{dx}+fv\right)dx - \cancel{v}^{\,0}\, u\frac{du}{dx}\Big|_0^1$$

$$\boxed{0 = \int_0^1 \left(u\frac{du}{dx}\frac{dv}{dx}+fv\right)dx}$$

not symmetric, no quadratic functional.

Second Order Equations

Tuesday, September 10, 2024 11:38 AM

Steps:

- Discretization (representation of domain by elements)
- Interpolation (approximation over an element)
- Element equations (develop finite element models)
- Assembly of elements (global or assembled equations)
- Imposition of boundary conditions (condensed equations)
- Solution (solved for unknown primary variable(s))
- Computation of secondary equations

typical element

$$0 = \int_{x_A}^{x_B} \left(a_e \frac{dv}{dx}\frac{du}{dx} + c_e v u - v f_e \right) dx - Q_A v(x_A) - Q_B v(x_B)$$

nodal value

shape/interpolation function

$$u = \sum_{j=1}^{n} u_j^e \psi_j(x)$$

Combining,

$$0 = \int_{x_A}^{x_B} \left[a_e \frac{d\psi_i^e}{dx} \left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c_e \psi_i^e \left(\sum_{j=1}^{n} u_j^e \psi_j^e \right) - \psi_i^e f_e \right] dx - \psi_i^e(x_A) Q_A - \psi_i^e(x_B) Q_B$$

$$= \sum_{j=1}^{n} \left[\int_{x_A}^{x_B} \left(a_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) u_j^e - \psi_i^e f_e \right] dx - \psi_i(x_A) Q_A - \psi_i(x_B) Q_B$$

Element force vector

$$= \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e$$

element stiffness matrix

Ex: solve

$$-\frac{d}{dx}\left(a \frac{du}{dx} \right) - q = 0 \qquad 0 < x < L$$

$$u(0) = 0, \quad \left(a \frac{du}{dx} \right) \Big|_{x=L} = Q_0$$

use two quadratic elements

given $a = 1$, $q = x$, $L = 1$, $Q_0 = 0$

Model Eqn:

$$-\frac{d}{dx}\left(a \frac{du}{dx} \right) + cu - q = 0$$

$$K_{ij}^e = \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx$$

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

$$K_{ij}^e = \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e\right)dx$$

$$f_i^e = \int_0^{h_e} q_e \psi_i^e \, dx$$

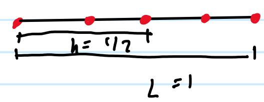
Element Eqn: $[K^e]\{u^e\} = \{f^e\} + \{\psi^e\}$

for quadratic,

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

0, $c = 0$, no c term in

$$\{f^e\} = \frac{q_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$ ← not applicable, $q = x$

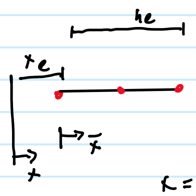


2 quadratic elements, 5 global nodes

since $L=1$ and uniform, $h_e = \frac{1}{2}$

$$K^1 = K^2 = \frac{1}{3(\frac{1}{2})}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}$$

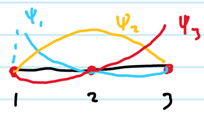
element force vector, $\{f^e\}$



$x = \bar{x} + x_e$

$$f_i^e = \int_0^{h_e} q_e(\bar{x})\, \psi_i^e(\bar{x})\, d\bar{x}$$

$$= \int_0^{h_e} (\bar{x} + x_e)\, \psi_i^e(\bar{x})\, d\bar{x}$$



$$f_1^e = \int_0^{h_e} (\bar{x} + x_e)\left(1 - \frac{\bar{x}}{h_e}\right)\left(1 - \frac{2\bar{x}}{h_e}\right)dx$$

$$= \left(\frac{h_e^2}{2} - h_e^2 + \frac{h_e^2}{\sim}\right) + x_e\left(h_e - \frac{3}{2}h_e + \frac{2}{3}h_e\right)$$

$$= \frac{x_e h_e}{6}$$

$$f_2^e = \int_0^{h_e} (\bar{x} + x_e)\, \overbrace{4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h_e}\right)}^{\psi_2(\bar{x})} d\bar{x}$$

$$= 2\left(\frac{h_e^2}{6} + x_e \frac{h_e}{3}\right)$$

$$f_3^e = \int_0^{h_e} (\bar{x} + x_e)\, \overbrace{\left[-\frac{\bar{x}}{h_e}\left(1 - \frac{2\bar{x}}{h_e}\right)\right]}^{\psi_3(\bar{x})} d\bar{x}$$

$$= -\left[\frac{h_e^2}{3} - \frac{h_e^2}{2} + x_e\left(\frac{h_e}{2} - \frac{2}{3}h_e\right)\right]$$

$$= \frac{h_e^2}{6} + \frac{x_e h_e}{6}$$

element 1:

$x_e = 0$

$h_1 = \frac{1}{2}$

$\{f^1\} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix} \cdot \begin{Bmatrix} 0 \\ \ldots \end{Bmatrix}$

$$\{f^1\} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/12 \\ 1/24 \end{Bmatrix}$$

Element 2:

$x_e = 1/2$

$h_2 = 1/2$

$$f^2 = \begin{Bmatrix} 1/24 \\ 1/6 \\ 1/12 \end{Bmatrix}$$

Equations Assemble!

$$K^1 = \frac{1}{3} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

$$K^2 = \frac{1}{3} \begin{array}{c} \begin{array}{ccc} u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix} \end{array} \begin{array}{c} u_3 \\ u_4 \\ u_5 \end{array}$$

$$K = \frac{1}{3} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & (14+14) & -16 & 2 \\ 0 & 0 & -16 & 32 & -16 \\ 0 & 0 & 2 & -16 & 14 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}$$

global/Assembled Eqn:

$$\frac{1}{3} \begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & (14+14) & -16 & 2 \\ 0 & 0 & -16 & 32 & -16 \\ 0 & 0 & 2 & -16 & 14 \end{bmatrix} \begin{Bmatrix} \cancel{u_1}^{\,0} \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/12 \\ 1/24 + \frac{1}{24} \\ 1/6 \\ 1/12 \end{Bmatrix} + \begin{Bmatrix} \psi_1^1 \\ \psi_2^1 \\ \psi_3^1 + \psi_1^2 \\ \psi_2^2 \\ \psi_3^2 \end{Bmatrix}$$

Condensed Eqn:

$$\frac{1}{3} \begin{bmatrix} 32 & -16 & 0 & 0 \\ -16 & (14+14) & -16 & 2 \\ 0 & -16 & 32 & -16 \\ 0 & 2 & -16 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{1}{12} \begin{Bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{Bmatrix}$$

$u_1 = 0$

$u_2 = .1224$

$u_3 = .02292$

$u_4 = .3047$

$u_5 = .733$

$$\frac{14}{3} u_1 - \frac{16}{3} u_2 + \frac{2}{3} u_3 + 0 u_4 + 0 u_5 = Q_1^1$$

$$Q_1^1 = -.05$$

Homework 2

Thursday, September 12, 2024 11:02 AM

C-Homewor
k+2+

Due: September 19, 2024 NAME: Easton Ingram

HOMEWORK SET # 2
ME/AE 5212 Introduction to Finite Element Analysis

1. (# 2.11 Text)
   Use trigonometric functions for the two-parameter Ritz approximation and obtain the Ritz coefficients using <u>quadratic functional approach</u> for the following equation:

$$-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right)=0 \quad \text{for } 0<x<1$$

$$u(0)=0,\ u(1)=1$$

   Compare with the exact solution $u_0 = \frac{\log_{10}(1+x)}{\log_{10}(2)}$ at x=1/2.

   Given: $u = \phi_0 + c_1\phi_1 + c_2\phi_2$

$$\phi_0 = \sin\frac{\pi x}{2},\ \phi_1 = \sin \pi x,\ \phi_2 = \sin 2\pi x$$

(10 points)

2. (# 2.24 Text)

   Consider the differential equation

$$-\frac{d^2u}{dx^2}=\cos \pi x \qquad \text{for } 0<x<1$$

   subject to the boundary condition

$$u(0)=0,\ u(1)=0$$

   Determine a three-parameter solution, with trigonometric functions using <u>collocation</u> at $x=\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.
   Compare with exact solution $u_0 = \pi^{-2}(cos\pi x + 2x - 1)$ at x=1/4.
   Given: $u = \phi_0 + c_1\phi_1 + c_2\phi_2 + c_3\phi_3$

$$\phi_0 = 0,\ \phi_1 = \sin \pi x,\ \phi_2 = \sin 2\pi x,\ \phi_3 = \sin 3\pi x$$

(10 points)

Weak form:

$$0 = \int_0^1 (1+x) \frac{dv}{dx} \frac{du}{dx}\, dx$$

$$= B(u,v) - \cancel{\ell(v)}^{\,0}$$

$$B(u,v) = B(v,u) \qquad \text{Bilinear symmetric}$$

Quadratic functional:

$$I(u) = \frac{1}{2} B(u,v) - \cancel{\ell(v)}^{\,0}$$

$$I = \frac{1}{2} \int_0^1 (1+x)\left(\frac{du}{dx}\right)^2 dx$$

$$u = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$= c_1 \sin \pi x + c_2 \sin 2\pi x + \sin \frac{\pi x}{2}$$

$$\frac{du}{dx} = c_1 \pi \cos \pi x + c_2 2\pi \cos 2\pi x + \frac{\pi}{2} \cos \frac{\pi x}{2}$$

```
clc
clear all

format long

syms x c1 c2

u=c1*sin(pi*x)+c2*sin(2*pi*x)+sin(pi*x/2)
du=diff(u,x)
du_2=du^2

I=(1/2)*int((1+x)*du_2,x, 0, 1)

[c1_sol, c2_sol] = solve([diff(I, c1) == 0, diff(I, c2) == 0], [c1, c2])

c1_sol = double(c1_sol)
c2_sol = double(c2_sol)

u_sol = subs(u, [c1, c2], [c1_sol, c2_sol])

ritz_sol = subs(u_sol, x, 1/2)
ritz_sol = double(ritz_sol)

exact_sol=double(subs(log10(1+x)/log10(2), x,1/2))
```

```
du =

(pi*cos((pi*x)/2))/2 + c1*pi*cos(pi*x) + 2*c2*pi*cos(2*pi*x)

du_2 =

((pi*cos((pi*x)/2))/2 + c1*pi*cos(pi*x) + 2*c2*pi*cos(2*pi*x))^2

I =

(3*c1^2*pi^2)/8 - (20*c1*c2)/9 + (2*c1*pi)/3 - (10*c1)/9 + (3*c2^2*pi^2)/2 - (4*c2*pi)/15 - (68*c2)/225 + (3*pi^2)/32 - 1/8

c1_sol =

(8*(- 405*pi^3 + 675*pi^2 + 120*pi + 136))/(5*(729*pi^4 - 1600))

c2_sol =

(4*(405*pi^3 + 459*pi^2 - 3000*pi + 5000))/(25*(729*pi^4 - 1600))

c1_sol =

  -0.124073713623187

c2_sol =

   0.029189312440156

u_sol =

(4206623252118049*sin(2*pi*x))/144115188075855872 - (2235113321759437*sin(pi*x))/18014398509481984 + sin((pi*x)/2)

ritz_sol =

2^(1/2)/2 - 2235113321759437/18014398509481984

ritz_sol =

   0.583033067563360

exact_sol =

   0.584962500721156
```

2.

$$R = -\frac{d^2u}{dx} - \cos \pi x$$

$$u = c_1 \sin \pi x + c_2 \sin 2\pi x + c_3 \sin 3\pi x$$

$$\frac{du}{dx} = c_1 \pi \cos \pi x + c_2 2\pi \cos 2\pi x + c_3 3\pi \cos 3\pi x$$

$$\frac{d^2u}{dx^2} = -c_1 \pi^2 \sin \pi x - c_2 4\pi^2 \sin 2\pi x - c_3 9\pi^2 \sin 3\pi x$$

$$R = c_1 \pi^2 \sin \pi x + c_2 4\pi^2 \sin 2\pi x + c_3 9\pi^2 \sin 3\pi x - \cos \pi x$$

$$R\left(\tfrac{1}{4}\right) = 0$$

$$R\left(\tfrac{1}{2}\right) = 0$$

$$R\left(\tfrac{3}{4}\right) = 0$$

```
clc
clear all

format long

syms x c1 c2 c3

u=c1*sin(pi*x)+c2*sin(2*pi*x)+c3*sin(3* pi*x)
du=diff(u,x)
du2=diff(du,x)

R=-du2-cos(pi*x)

[c1_sol, c2_sol, c3_sol] = solve([subs(R, x, 1/4) == 0, subs(R, x, 1/2) == 0, subs(R, x, 3/4) == 0], [c1, c2, c3])
c1_sol = double(c1_sol)
c2_sol = double(c2_sol)
c3_sol = double(c3_sol)

u_sol = subs(u, [c1, c2, c3], [c1_sol, c2_sol, c3_sol])

collocation_sol = subs(u_sol, x, 1/4);
collocation_sol = double(collocation_sol)

exact_sol=double(subs(pi^(-2)*(cos(pi*x)+2*x-1), x, 1/4))
```

```
du2 =

- c1*pi^2*sin(pi*x) - 4*c2*pi^2*sin(2*pi*x) - 9*c3*pi^2*sin(3*pi*x)


R =

c1*pi^2*sin(pi*x) - cos(pi*x) + 4*c2*pi^2*sin(2*pi*x) + 9*c3*pi^2*sin(3*pi*x)


c1_sol =

0


c2_sol =

2^(1/2)/(8*pi^2)


c3_sol =

0


c1_sol =

     0


c2_sol =

   0.017911224007836


c3_sol =

     0


u_sol =

(5162558833116179*sin(2*pi*x))/288230376151711744


collocation_sol =

   0.017911224007836


exact_sol =

   0.020984304210176
```

Applications of 2nd Order FEA

Thursday, September 19, 2024 12:01 PM

- Solid mechanics
- Heat transfer
- Fluid Mechanics
- More in book

Solid mechanics

Hookers Law

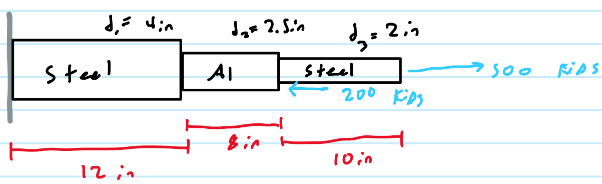
$\varepsilon_x = \frac{\partial u}{\partial x}$ - normal strain

$\varepsilon_y = \frac{\partial v}{\partial y}$ - normal strain

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ - shear strain

Ex: Axial deformation of a bar

$$-\frac{d}{dx}\left[EA\frac{du}{dx}\right] = 0 \qquad 0 < x < L$$



$E_s = 30\times10^6$ psi
$E_a = 10\times10^6$ psi

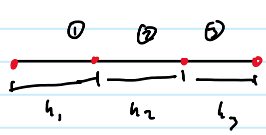
3 elements, 4 nodes, linear

Model Eqn:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$K^e = \frac{a_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \cancel{\frac{c_e h_e}{6}}^{\,0}\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}$$



$$\frac{a_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^e\\ u_2^e\end{Bmatrix} = \begin{Bmatrix}Q_1^e\\ Q_2^e\end{Bmatrix}$$

$e=1$: $\dfrac{E_e A_e}{h_e} = \dfrac{30\times10^6 \,\frac{\pi 4^2}{4}}{} = 120\pi \times 10^6$

. . .

$$
\begin{bmatrix}
\frac{E_1A_1}{h_1} & -\frac{E_1A_1}{h_1} & 0 & 0 \\
-\frac{E_1A_1}{h_1} & \frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} & 0 \\
0 & -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2}+\frac{E_3A_3}{h_3} & -\frac{E_3A_3}{h_3} \\
0 & 0 & -\frac{E_3A_3}{h_3} & \frac{E_3A_3}{h_3}
\end{bmatrix}
\begin{Bmatrix} \cancel{u_1} \to 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}
=
\begin{Bmatrix} Q_1^1 \\ \cancel{Q_2^1+Q_1^2} \to 0 \\ \cancel{Q_2^2+Q_1^3} \to -200\times10^3 \\ \cancel{Q_2^3} \to 500\times10^3 \end{Bmatrix}
$$

Condensed Eqn:

$$
\begin{bmatrix}
\frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} & 0 \\
-\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2}+\frac{E_3A_3}{h_3} & -\frac{E_3A_3}{h_3} \\
0 & -\frac{E_3A_3}{h_3} & \frac{E_3A_3}{h_3}
\end{bmatrix}
\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ -200\times10^3 \\ 500\times10^3 \end{Bmatrix}
$$

solving, $u_2 = .95493 \times 10^{-3}$ in

$u_3 = .58442 \times 10^{-1}$ in

$u_4 = 0.1115$ in

$$\frac{E_1A_1}{h_1}\cancel{u_1}^{\,0} - \frac{E_1A_1}{h_1}u_2 + 0u_3 + 0u_4 = Q_1^1$$

$$\frac{-120\pi(10^6)}{12}\left(.95493\times10^{-3}\right) = Q_1^1$$

$$Q_1^1 = -299.5 \text{ kips}$$

Heat Transfer

- conduction

  Fourier's Law:

  $$q = -KA\frac{dT}{dx}$$

- convection

  Newton's Law of Cooling:

  α or h

Newton's law of cooling:

$$q = \overset{\beta \text{ or } h}{\beta} A\,(T_s - T_\infty)$$

- radiation

Stefan-Boltzman law:

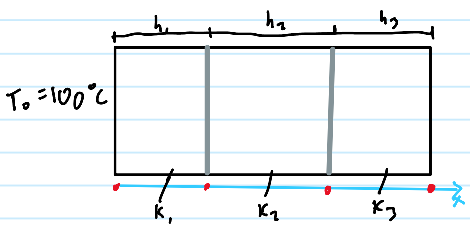
$$q = \varepsilon \sigma T_s^4$$

$\uparrow$ $5.67 \times 10^{-8}\ W/m^2 \cdot K^4$

Ex: Nodal temperature field in composite wall, find heat flux at node 1.

$$-\frac{d}{dx}\left[kA\frac{dT}{dx}\right] = 0 \qquad 0 < x < L$$

BC: $\quad T_{x=0} = T_0 \qquad \left[kA\frac{dT}{dx} + \beta A(T - T_\infty)\right]_{x=L} = 0$



$T_\infty = 35°C$

$\beta = 15\ W/m^2 \cdot °C$

$k_1 = 50\ W/m \cdot °C$

$k_2 = 30\ W/m \cdot °C$

$k_3 = 70\ W/m \cdot °C$

$h_1 = 50 \times 10^{-3}\ m$

$h_2 = 35 \times 10^{-3}\ m$

$h_3 = 25 \times 10^{-3}\ m$

$A = 1m^2$

3 elements
4 global nodes

linear elements

Model eqn:

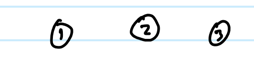
$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

linear element: $[K^e]\{u^e\} = \{f^e\} + \{\psi^e\}$

$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\{f^e\} = \frac{q_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

where $a_e = k_e A_e$, $c_e = 0$, $q_e = 0$



$$\frac{k_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} \psi_1^e \\ \psi_2^e \end{Bmatrix}$$

Element 1:

$$K^1 = \frac{K_1 A}{h_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{50\,(1)}{50\times10^{-3}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 1000\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

E 2:

$$K^2 = \frac{30\,(1)}{35\times10^{-3}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 857.1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

3:

$$K^3 = \frac{70\,(1)}{25\times10^{-3}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 2800\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Assembled / Global Eqn:

$$\begin{bmatrix}1000 & -1000 & 0 & 0\\ -1000 & 1000+857.1 & -857.1 & 0\\ 0 & -857.1 & 857.1+2800 & -2800\\ 0 & 0 & -2800 & 2800\end{bmatrix}\begin{Bmatrix}u_1 \rightarrow 100 \text{ (non-homogeneous)}\\ u_2\\ u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}Q^1_1\\ \cancel{Q^1_2+Q^2_1} \rightarrow 0\\ \cancel{Q^2_2+Q^3_1} \rightarrow 0\\ \cancel{Q^3_2} \rightarrow \beta_R\,(u_4-u_\infty)\end{Bmatrix}$$

$$\begin{bmatrix}1000 & -1000 & 0 & 0\\ -1000 & 1857.1 & -857.1 & 0\\ 0 & -857.1 & -3657.1 & 2800\\ 0 & 0 & 2800 & 2800+\beta_R\end{bmatrix}\begin{Bmatrix}100\\ u_2\\ u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}Q^1\\ 0\\ 0\\ \beta_R u_\infty\end{Bmatrix}$$

condensed eqn:

$$\begin{bmatrix}1857.1 & -857.1 & 0\\ -857.1 & -3657.1 & 2800\\ 0 & 2800 & 2800+\beta_R\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}100{,}000\\ 0\\ 525\end{Bmatrix}\begin{matrix}\leftarrow K^1_{12}u_1 \quad (1000)(100)\\ \\ \leftarrow 15(35)\end{matrix}$$

solving,
$$\boxed{\begin{aligned}u_2 &= 99.06\,^\circ C\\ u_3 &= 97.96\,^\circ C\\ u_4 &= 97.63\,^\circ C\end{aligned}}$$

$$1000(100) - 1000\,u_2 + 0\,u_3 + 0\,u_4 = Q^1_1$$

$$Q^1_1 = 940 \text{ W/m}^2$$

Fluid Mechanics:

$$R_e = \frac{\rho V D}{\mu}$$

$R_e > 2400$, turbulent

$2100 < R_e < 2400$, transition region

$R_e < 2100$, laminar

$$Q = \frac{\pi D^4 \Delta P}{128\,\mu L}$$ Laminar

$$Q = \frac{\pi D^4 \Delta P}{128 \mu L} \quad \text{Laminar}$$

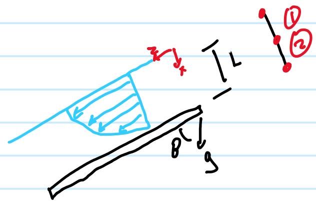
Stream Function

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

Darcy's Law :

$$u = -\frac{k}{\mu} \frac{\partial P}{\partial x}$$

Ex: Momentum Eqn along z :



$$-\mu \frac{d^2 w}{dx^2} = \rho g \cos\theta$$

incompressible, laminar flow

BC: $\left.\frac{dw}{dx}\right|_{x=0} = 0$, $w(x=L) = 0$

$$T_{xz} = -\mu \frac{dw}{dx}$$

use two linear elements

Exact solution:

$$w_e = \frac{\rho g L^2 \cos\theta}{2\mu}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

modeling eqn:

$$-\frac{d}{dx}\left(a \frac{du}{dx}\right) + cu - q = 0$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

constant a_e, c_e, q_e, h_e

$$K^e = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$f^e = \frac{q_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$q = \rho g \cos\theta$, $a = \mu$, $c = 0$, $h_1 = h_2 = h_e = \frac{L}{2}$

$$f^e = \frac{q_e h_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$q = \rho g \cos\theta$, $a = \mu$, $c = 0$, $h_1 = h_2 = h_e = \frac{L}{2}$

$$[K^e] = \frac{\mu}{h}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = K^1 = K^2$$

Assembled Eqn:

$$\frac{\mu}{h}\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\\cancel{u_3}\to 0\end{Bmatrix} = \frac{\rho g h\cos\theta}{2}\begin{Bmatrix}1\\2\\1\end{Bmatrix} + \begin{Bmatrix}\cancel{Q_1^1}\to 0\\\cancel{Q_2^1 + Q_1^2}\to 0\\Q_2^2\end{Bmatrix}$$

Condensed eqn:

Delete last row and column

$$\frac{\mu}{h}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \frac{\rho g h\cos\theta}{2}\begin{Bmatrix}1\\2\end{Bmatrix}$$

Solving,

$$\boxed{u_1 = \frac{\rho g L^2\cos\theta}{2\mu} \qquad u_2 = \frac{3\rho g L^2\cos\theta}{8\mu}}$$

nodal velocities

Velocity at any point, $u(x)$:

Element 1: $u(x) = u_1^1\psi_1^1 + u_2^1\psi_2^1 = u_1\psi_1^1 + u_2\psi_2^1 \qquad 0 < x < \frac{L}{2}$

Element 2: $u(x) = u_1^2\psi_1^2 + u_2^2\psi_2^2 = u_2\psi_1^2 + u_3\psi_2^2 \qquad \frac{L}{2} < x < L$

$e = 1$:

$$u(x) = u_1\left(1 - \frac{\bar{x}}{h_e}\right) + u_2\left(\frac{\bar{x}}{x_e}\right)$$

$x_e = 0$, $h_e = L/2$

$$u(x) = u_1\left(1 - \frac{x}{h_e}\right) + u_2\left(\frac{x}{h_e}\right)$$

$$= \frac{\rho g L^2\cos\theta}{2\mu}\left[1 - \frac{x}{L/2}\right] + \frac{3\rho g L^2\cos\theta}{8\mu}\left(\frac{x}{L/2}\right)$$

$$\boxed{= \frac{\rho g L^2\cos\theta}{2\mu}\left[1 - \frac{x}{2L}\right]}$$

$e = 2$:

$$u(x) = u_2\psi_1^2 + \cancel{u_3}\to 0\,\psi_2^2$$

$$= \frac{3\rho g L^2\cos\theta}{8\mu}\left[1 - \frac{\bar{x}}{h_e}\right]$$

$$u(x) = u_2 \, \Psi_1 + \cancel{u_3} \, \Psi_2$$

$$= \frac{3\rho g L^2 \cos\theta}{8\mu} \left[1 - \frac{\bar{x}}{h_e}\right]$$

$$\bar{x} = x - x_e = x - \frac{L}{2}$$

$$= \frac{3\rho g L^2 \cos\theta}{8\mu} \left[1 - \frac{x - L/2}{L/2}\right]$$

$$\Rightarrow \boxed{\frac{3\rho g L^2 \cos\theta}{8\mu} \left(1 - \frac{x}{L}\right)}$$

Exact solution:

$$w_e = \frac{\rho g L^2 \cos\theta}{2\mu} \left[1 - \left(\frac{x}{L}\right)^2\right]$$

Compare at nodal values.

$$w_e\big|_{x=0} = \frac{\rho g L^2 \cos\theta}{2\mu} \qquad \text{matches}$$

$$w_e\big|_{x=L/2} = \frac{\rho g L^2 \cos\theta}{2\mu} \left[\frac{3}{4}\right] \qquad \text{matches}$$

$$w_e\big|_{x=L} = 0 \qquad \text{matches}$$

varies between nodes

Evaluate shear stress:

Velocity Field.

Exact

$$T_{xz} = -\mu \frac{dw}{dx}$$

$$= -\mu \frac{\rho g L^2 \cos\theta}{2\mu} \left(-\frac{2x}{L^2}\right)$$

$x = L$: $T_{xz} = \rho g L \cos\theta$

Equilibrium:

use last of assembled eqn

$$\frac{\mu}{h}(0)\,\cancel{u_1} + \frac{\mu}{h}(-1)\,u_2 + \frac{\mu}{h}(1)\,\cancel{u_3} = \frac{\rho g h \cos\theta}{2} + Q_2^2$$

$$\Rightarrow Q_2^2 = -\rho g L \cos\theta$$

$x = L$: $T_{xz} = -Q_2^2 = \rho g L \cos\theta$ matches exactly

Homework 3

Thursday, September 26, 2024 11:07 AM

C-Homewor
k+3

Due Date: October 1, 2024 NAME: Easton Ingram

HOMEWORK SET # 3
ME/AE 5212 Introduction to Finite Element Analysis

1. Solve the following differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-q=0 \qquad 0<x<L$$

for the boundary conditions

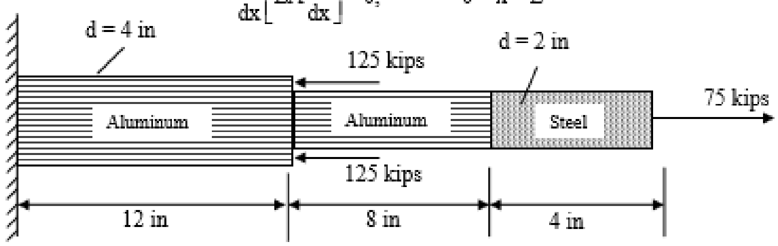
$$u(0)=0 \qquad \left(a\frac{du}{dx}\right)\Big|_{x=L}=Q_0$$

Use four linear elements. Assume a = 1, q = x, L = 2 and Q_o=0

(10 points)

2. Determine the unknown displacements of the stepped bar. Use the minimum number of linear bar elements.

$$-\frac{d}{dx}\left[EA\frac{du}{dx}\right]=0, \qquad 0<x<L$$



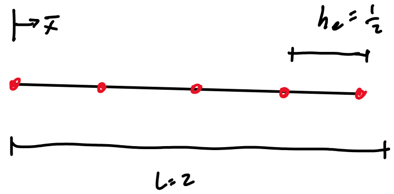
Steel, E_s=30 x 10^6 psi, Aluminum, E_a = 10 x 10^6 psi

(10 points)

1.

Modeling Eqn:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + \cancel{c}^{0}u - q = 0$$



$h_e = \frac{1}{2}$ 4 linear elements, 5 nodes

Element Eqn:

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$K^e = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f_i^e = \int_0^{h_e} \psi^e q_e \, d\bar{x}$$

$$f_1^e = \int_0^{h_e} (\bar{x} + x_e)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x}$$

$$= \int_0^{h_e} \left(\bar{x} - \frac{\bar{x}^2}{h_e} + x_e - \frac{x_e \bar{x}}{h_e}\right) d\bar{x}$$

$$= \frac{1}{2}\bar{x}^2 - \frac{\bar{x}^3}{3h_e} + x_e\bar{x} - \frac{x_e \bar{x}^2}{2h_e}\Bigg|_{h_e}$$

$$= \frac{1}{2}h_e^2 - \frac{1}{3}h_e^2 + x_e h_e - \frac{x_e h_e}{2}$$

$$= \underline{\frac{1}{6}h_e^2 \cdot \frac{1}{2}x_e h_e}$$

$$f_2^e = \int_0^{h_e} (\bar{x} + x_e)\left(\frac{\bar{x}}{h_e}\right) d\bar{x} = \int_0^{h_e}\left(\frac{\bar{x}^2}{h_e} + \frac{\bar{x}x_e}{h_e}\right) d\bar{x}$$

$$= \frac{\bar{x}^3}{3h_e} + \frac{\bar{x}^2 x_e}{2h_e}\Bigg|_{h_e}$$

$$= \underline{\frac{h_e^2}{3} + \frac{h_e}{2}x_e}$$

$e = 1$: $x = 0$ $h_1 = 1/2$

$$f^1 = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{24} \\ \frac{1}{12} \end{Bmatrix}$$

$e = 2$: $x_2 = \frac{1}{2}$ $h_2 = 1/2$

$$f^2 = \begin{Bmatrix} -\frac{1}{12} \\ \frac{5}{24} \end{Bmatrix}$$

$e=3$: $\quad x_3 = 1 \quad h_3 = \frac{1}{2}$

$$F^3 = \begin{Bmatrix} -\frac{5}{24} \\ \frac{1}{3} \end{Bmatrix}$$

$e=4$: $\quad x_4 = \frac{3}{2} \quad h_4 = \frac{1}{2}$

$$F^4 = \begin{Bmatrix} -\frac{1}{3} \\ \frac{5}{12} \end{Bmatrix}$$

Assembled Eqn:

$$K_e = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K = 2\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Global eqn:

$$2\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 + f_1^4 \\ f_2^4 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

$$\frac{1}{24}$$

$$\frac{1}{12} + -\frac{1}{2}$$

$$\frac{5}{24} + -\frac{5}{24}$$

$$\frac{1}{3} + -\frac{1}{3}$$

$$\frac{5}{12}$$

$$2\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{u_1} \to 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{24} \\ 0 \\ 0 \\ 0 \\ \frac{5}{12} \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ \cancel{Q_2^1 + Q_1^2} \to 0 \\ \cancel{Q_2^2 + Q_1^3} \to 0 \\ \cancel{Q_2^3 + Q_1^4} \to 0 \\ \cancel{Q_2^4} \to 0 \end{Bmatrix}$$

$Q_5 = 0$

Condensed:

$$2\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{12} \end{Bmatrix}$$

Solve via MATLAB:

$$\boxed{\begin{aligned} u_2 &= -5/24 \\ u_3 &= -5/12 \\ u_4 &= -5/24 \\ u_5 &= 0 \end{aligned}}$$

$$Q_1^1 = 2\cancel{u_1}^{0} - 2u_2 = Q_1^1$$

$$\boxed{Q_1^1 = 5/12}$$

2

$$[k^e]\{u^e\} = \{\cancel{f^e}^{\,0,\text{ no distributed load}}\} + \{Q^e\}$$

$$[k^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{a^e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

$e=1$: $E_1A_1 = 10\times10^6\left(\frac{\pi\, 4^2}{4}\right) = 40\pi\times10^6$, $h_1 = 12$, $d_1 = 4$

$e=2$: $E_2A_2 = 10\times10^6\left(\pi\,\frac{2^2}{4}\right) = 10\pi\times10^6$, $h_2 = 8$, $d_2 = 2$

$e=3$: $E_3A_3 = 30\times10^6\left(\pi\,\frac{2^2}{4}\right) = 30\pi\times10^6$, $h_3 = 4$, $d_3 = 2$

Assembled Eqn:

$$\begin{bmatrix} \frac{E_1A_1}{h_1} & -\frac{E_1A_1}{h_1} & 0 & 0 \\ -\frac{E_1A_1}{h_1} & \frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} & 0 \\ 0 & -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2}+\frac{E_3A_3}{h_3} & -\frac{E_3A_3}{h_3} \\ 0 & 0 & -\frac{E_3A_3}{h_3} & \frac{E_3A_3}{h_3} \end{bmatrix}\begin{Bmatrix} \cancel{u_1}^{\,0} \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1+Q_1^2 \to -250\times10^3 \\ Q_2^2+Q_1^3 \to 0 \\ Q_2^3 \to 75\times10^3 \end{Bmatrix}$$

Condensed Eqn:

$$\begin{bmatrix} \frac{E_1A_1}{h_1} + \frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} & 0 \\ -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2} + \frac{E_3A_3}{h_3} & -\frac{E_3A_3}{h_3} \\ 0 & -\frac{E_3A_3}{h_3} & \frac{E_3A_3}{h_3} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -250\times10^3 \\ 0 \\ 75\times10^3 \end{Bmatrix}$$

$$10^6 \begin{bmatrix} 4.583 & -1.25 & 0 \\ -1.25 & 8.75 & -7.5 \\ 0 & -7.5 & 7.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -250\times10^3 \\ 0 \\ 75\times10^3 \end{Bmatrix}$$

Solving,

$$\boxed{\begin{aligned} u_2 &= -\ 0602 \\ u_3 &= -.0002 \\ u_4 &= 0\ 0048 \end{aligned}}$$

A-Midterm+Exam-FEA

Tuesday, October 8, 2024 10:57 AM

A-Midterm
+Exam-FEA

NAME: Easton Ingram

ME/AE 5212
INTRODUCTION TO FINITE ELEMENT ANALYSIS
MIDTERM EXAM

Guidelines:

1. Exam is open book (Class notes, HW and textbook)
2. Need handwritten solution with all the steps and only calculator is allowed.
3. The test must be completed individually **in one sitting.**
4. Exam duration is 1 hr 30 min (11 am – 12:30 pm).
5. Submit the solution along with this cover page (with full name and signature) as a **single pdf file** in canvas.

Missouri S&T Student Academic Conduct and Honor Code Statement:

I affirm that I have not given or received any unauthorized help on this exam, and that this is my own work.

SIGNATURE Easton Ingram

1. Find a one-parameter <u>Least Squares</u> solution of the equation

$$-\frac{d^2u}{dx^2} + x^3 = 0 \text{ for } \quad 0 < x < 1$$
$$u(0) = 0, \left(\frac{du}{dx}\right)_{x=1} = 4$$

Use $\phi_0 = 2x$ and $\phi_1 = x\,(x-2)$

(20 points)

2. Construct the weak form of the following equation and express it as bilinear and linear functions. Also, obtain the quadratic functional if possible.

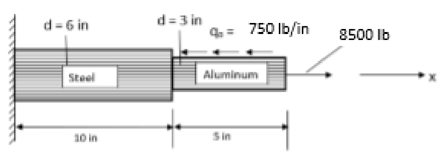
$$-\frac{d}{dx}\left[(1+2x^2)\frac{du}{dx}\right] + u = x^3 \quad \text{for } 0 < x < 1$$

$$u(1) = 8, \qquad \left(\frac{du}{dx}\right)_{x=0} = 16$$

(20 points)

3. For the composite bar (circular cross-section with diameter d) shown in figure, determine the axial displacements. Also, determine the secondary unknown (reaction force). Use $E_{St} = 30 \times 10^6$ psi, $E_{Al} = 10 \times 10^6$ psi, and the minimum number of linear elements.

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \quad \text{for } 0 < x < L$$



(30 points)

4. An insulating wall is constructed of three homogeneous layers with conductivities k_1, k_2 and k_3 in intimate contact. Under steady-state conditions, the temperatures of the media in contact at the left and right surfaces of the wall are at ambient temperature of T_∞^L and T_∞^R, respectively, and film coefficients β_L and β_R, respectively. Assume that there is no internal heat generation and that the heat flow is one-dimensional $(\partial T/\partial y = 0)$. Use the minimum number of linear finite elements to solve the problem. Determine the **simplified condensed equations only** (need not solve for temperatures).

The governing equation is

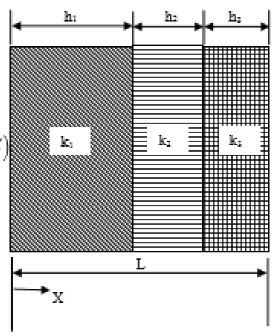
$$-\frac{d}{dx}\left[kA\frac{dT}{dx}\right]=0 \qquad 0<x<L$$

and boundary conditions are:

$$\left[-kA\frac{dT}{dx}+\beta A(T-T_\infty)\right]_{x=0}=0; \quad \left[kA\frac{dT}{dx}+\beta A(T-T_\infty)\right]_{x=L}=0$$

$T_\infty^L = 140°$ C

$\beta_L = 10W/(m^2\cdot°C)$



Air at temperature,

$T_\infty^R = 35$ ° C

Film coefficient,

$\beta_R = 15W/(m^2\cdot°C)$

$k_1 = 80$ W/(m . °C)

$k_2 = 40$ W/(m . °C)

$k_3 = 20$ W/(m . °C)

$A = 1\ m^2$

$h_1 = 0.075$ m

$h_2 = 0.040$ m

$h_3 = 0.035$ m

(30 points)

1.

$$\int_0^1 \frac{\partial R}{\partial c_1} R\, dx = 0$$

$$u = \phi_0 + c_1\phi_1 = 2x + c_1 x(x-2) = 2x + c_1x^2 - 2c_1x$$

$$R = -\frac{d^2u}{dx^2} + x^3$$

$$\frac{du}{dt} = 2 + c_1 2x \quad -2c_1$$

$$\frac{d^2u}{dt^2} = 2c_1$$

$$R = -2c_1 + x^3$$

$$\int_0^1 \frac{\partial R}{\partial c_1} R\,dx = \int_0^1 -2(-2c_1 + x^3)\,dx$$

$$= 4c_1 x - \frac{1}{4}x^4 \Big|_0^1$$

$$= 4c_1 - \frac{1}{4}$$

$$4c_1 - \frac{1}{4} = 0$$

$$c_1 = \frac{1}{4 \cdot 4}$$

$$c_1 = \frac{1}{16}$$

$$\boxed{u = 2x + \frac{1}{16}x(x-2)}$$

2. $-\frac{d}{dx}\left[(1+2x^2)\frac{du}{dx}\right] + u - x^3 = 0$

$$0 = \int_0^1 v\left[-\frac{d}{dx}\left[(1+2x^2)\frac{du}{dx}\right] + u - x^3\right] dx$$

$v = \delta u \Rightarrow v(1) = 0$

$$= \int_0^1 (1+2x^2)\frac{dv}{dx}\frac{du}{dx} - uv - vx^3\,dx - v(1+2x^2)\frac{du}{dx}\Big|_0^1$$

$$\boxed{0 = \int_0^1 \left[(1+2x^2)\frac{dv}{dx}\frac{du}{dx} - uv\right]dx + \int_0^1 vx^3\,dx + v(0)(1+2x^2)(16)}$$

$$B = \int_0^1 \left[(1+2x^2)\frac{du}{dx}\frac{dv}{dx} - uv\right]dx$$

$$l = \int_0^1 vx^3\,dx + v(0)(1+2x^2)(16)$$

billinear and symmetric.

$$I(u) = \frac{B(u,v)}{2} - l(u)$$

$$\boxed{I(u) = \frac{1}{2}\int_0^1\left[(1+2x^2)\left(\frac{du}{dx}\right)^2 - u^2\right]dx + \int_0^1 ux^3\,dx + u(0)(1+2x^2)(16)}$$

3. 2 linear elements, 3 nodes

Model Eqn:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

$$[k^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$k^e = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \cancel{\frac{c_e h_e}{6}}^{\,0}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$\Big[\ E_1 A_1 \quad -E_1 A_1 \quad \ldots$

[illegible]

$$\begin{bmatrix} \frac{E_1A_1}{h_1} & -\frac{E_1A_1}{h_1} & 0 \\ -\frac{E_1A_1}{h_1} & \frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} \\ 0 & -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2} \end{bmatrix} \begin{Bmatrix} \cancelto{0}{u_1} \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \cancelto{0}{f_1^1} \\ \cancelto{750}{f_2^1+f_1^2} \\ \cancelto{0}{f_2^2} \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ \cancelto{0}{Q_2^1+Q_1^2} \\ \cancelto{8500}{Q_2^2} \end{Bmatrix}$$

$$\frac{E_1A_1}{h_1} = \frac{30\times10^6 (6)^2\pi}{4(10)} = 8.482\times10^7$$

$$\frac{E_2A_2}{h_2} = \frac{10\times10^6 (3)^2\pi}{4(5)} = 1.414\times10^7$$

$$\frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} = 9.896\times10^7$$

$$10^7\begin{bmatrix} 8.482 & -8.482 & 0 \\ -8.482 & 9.896 & -1.414 \\ 0 & -1.414 & 1.414 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 750 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ 8500 \end{Bmatrix}$$

Condensed eqn

$$10^7\begin{bmatrix} 9.896 & -1.414 \\ -1.414 & 1.414 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 750 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 8500 \end{Bmatrix}$$

Solving, $\boxed{\begin{matrix} u_2 = .000109 \\ u_3 = .00071 \end{matrix}}$

$$\frac{E_1A_1}{h_1}\cancelto{0}{u_1} - \frac{E_1A_1}{h_1}u_2 + 0u_3 = Q_1^1$$

$$Q_1 = -\frac{E_1A_1}{h_1}u_2 = -8.482\times10^7(.000109) = \boxed{-9205.38 \text{ lbs}}$$

d.

Model eqn:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

$$K^e = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{3 linear elements, 4 nodes}$$

K^e $\frac{a^e}{\cdot}$ [1 -1] {u_1^e} {Q_1^e}

$$\frac{k^e A^e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^e\\ u_2^e\end{Bmatrix}=\begin{Bmatrix}Q_1^e\\ Q_2^e\end{Bmatrix}$$

$e=1$:

$$k^1=\frac{k_1 A}{h_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{80\,(1)}{.075}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$=1067\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$e=2$:

$$k^2=2000\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$e=3$.

$$k^3=571\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Assembled global eqn:

$$\begin{bmatrix}1067 & -1067 & 0 & 0\\ -1067 & 3067 & -2000 & 0\\ 0 & -2000 & 2571 & -571\\ 0 & 0 & -571 & 571\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\ Q_2^1+Q_1^2\\ Q_2^2+Q_1^3\\ Q_2^3\end{Bmatrix}\begin{matrix}\beta_L(u_1-u_a)\\ 0\\ 0\\ \beta_R(u_4-u_a)\end{matrix}$$

Condensed Eqn:

$$\boxed{\begin{bmatrix}3067 & -2000 & 0\\ -200 & 2571 & -571\\ 0 & -571 & 571+\beta_R\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}1067(u_1)\\ 0\\ 525\end{Bmatrix}}$$

not sure about this

1

Computation

Tuesday, October 15, 2024 11:40 AM

Ex: Gauss Elimination method

$$x_1 - x_2 + 3x_3 = 10 \qquad 1$$

$$2x_1 + 3x_2 + x_3 = 15 \qquad 2$$

$$4x_1 + 2x_2 - x_3 = 6 \qquad 3$$

Eliminate x_1 from 2 by multiplying by -2 and adding 1 Same for 3 multiplied by -4

$$x_1 - x_2 + 3x_3 = 0$$
$$5x_2 - 5x_3 = -5$$
$$6x_2 - 13x_3 = -34$$

$$x_1 - x_2 + 3x_3 = 10$$
$$5x_2 - 5x_3 = -5$$
$$-7x_3 = -28$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & -5 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{matrix} 10 \\ -5 \\ -28 \end{matrix}$$

Upper triangular form

$$x_3 = 4$$
$$x_2 = 3$$
$$x_1 = 1$$

Numerical Integration:

Newton-Cotes quadrature

$$I = \int_a^b f(x)\,dx = (b-a)\sum_{i=1}^{r} \underbrace{w_i}_{\text{weights}} f(x)$$

Weight coefficients for the Newton-Cotes formula

r	w_1	w_2	w_3	w_4	w_5	w_6	w_7	
1	1							
2	$\frac{1}{2}$	$\frac{1}{2}$						← Trapezoidal rule
3	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					← Simpson's rule
4	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				
5	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			
6	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		
7	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	

Gauss - Legendre quadrature:

$$I = \int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{r} w_i f(\xi_i) = w_1 f(\xi_1) + w_2 f(\xi_2) + \ldots + w_r f(\xi_r)$$

Weights and Gauss points for the Gauss-Legendre quadrature

$$\int_{-1}^{1} F(\xi)\,d\xi = \sum_{i=1}^{r} F(\xi_i) w_i$$

Points, ξ_i†	r	Weights, w_i
0.0000000000	1	2.0000000000
±0.5773502692	2	1.0000000000
0.0000000000	3	0.8888888889
±0.7745966692		0.5555555555
±0.3399810435	4	0.6521451548
±0.8611363116		0.3478548451
0.0000000000	5	0.5688888889
±0.5384693101		0.4786286705
±0.9061798459		0.2369268850
±0.2386191861	6	0.4679139346
±0.6612093865		0.3607615730
±0.9324695142		0.1713244924

†Note that 0.57735....= $1/\sqrt{3}$, 0.77459...= $\sqrt{3/5}$, and 0.888...= 8/9, and 0.555...= 5/9

r+1 point Gauss quadrature provides Exact solution if polynomial is degree 2r+1 or less.

Ex:

$$I = \int_{1}^{3} (2x + 3x^2 + 4x^3)\,dx$$

Exact solution: $I = \left(x^2 + x^3 + x^4\right)\Big|_1^3 = 114$

using one point Gauss quadrature:

$w_1 = 2, \ \xi = 0$

$$I = \int_{-1}^{1} f(\xi)\,d\xi = w_1 f(\xi_1)$$

$$f(x) = 2x + 3x^2 + 4x^3$$

$$I = 2\left[2(\xi + 2) + 3(\xi + 2)^2 + 4(\xi + 2)^3\right]$$

$$= 2\left[4 + 12 + 32\right]$$

$$= \boxed{96}$$

two point quadrature.

$w_1 = w_2 = 1, \quad \xi_1 = \xi_2 = \frac{1}{\sqrt{3}}$

$$I = w_1 f(\xi_1) + w_2 f(\xi_2)$$

$I = w_1 f(\xi_1) + w_2 f(\xi_2)$

$= 1\left[2(\xi + 2) + 3(\xi + 2)^2 + 4(\xi + 2)^3\right]$

$= 2\left(\frac{1}{\sqrt{3}} + 2\right) + 3\left(\frac{1}{\sqrt{3}} + 2\right)^2 + 4\left(\frac{1}{\sqrt{3}} + 2\right)^3$

$= 114$ since $r = 2$ rule

Computer Project

Thursday, October 17, 2024 11:06 AM

E-Computer
-Project-2...

Beams

Thursday, October 17, 2024 12:01 PM

1-D:

Euler-Bernouli Beam theory
- No shear deformation
- Fourth order eqns
- Thin beam ($L/h < 100$)

Timoshenko Beam Theory
- Includes shear deformation
- Two second order eqns
- Moderately thick beam

Euler-Bernouli:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + k_f w = q$$

transverse deflection

distributed transverse load

modulus of elastic foundation

if $EI =$ const and $q =$ const and $k_f = 0$,

$$K^e = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

$$f^e = \frac{q_e h_e}{12}\begin{bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{bmatrix} + \begin{bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{bmatrix}$$

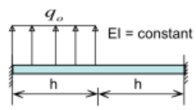
Assembled Equations:

$$[K]=\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 & 0 & 0 \\ K_{12}^1 & K_{22}^1 & K_{23}^1 & K_{24}^1 & 0 & 0 \\ K_{13}^1 & K_{23}^1 & K_{33}^1+K_{11}^2 & K_{34}^1+K_{12}^2 & K_{13}^2 & K_{14}^2 \\ K_{14}^1 & K_{24}^1 & K_{34}^1+K_{12}^2 & K_{44}^1+K_{22}^2 & K_{23}^2 & K_{24}^2 \\ 0 & 0 & K_{13}^2 & K_{23}^2 & K_{33}^2 & K_{34}^2 \\ 0 & 0 & K_{14}^2 & K_{24}^2 & K_{34}^2 & K_{44}^2 \end{bmatrix}$$

$$\{F\}=\begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1+q_1^2 \\ q_4^1+q_2^2 \\ q_3^2 \\ q_4^2 \end{Bmatrix}+\begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1+Q_1^2 \\ Q_4^1+Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

Example Problem 1 (# 5.8 Text)

Use the minimum number of Euler-Bernoulli beam finite elements to analyze the beam problem shown. Determine the unknown displacements and rotations.



Modeling Eqn:

$$[K^e]\{\Delta^e\} = \{F^e\}$$

$EI = \text{const}$, $q = \text{const}$, $K_f = 0$ ← no spring force

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

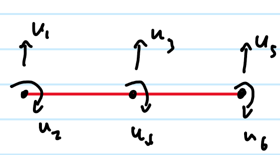
$$\{F^e\} = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$[K^1] = [K^2] = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3 \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$ 2 elements

$$\{F^1\} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

$$F^2 = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}.$$

Assemble.



2 DOF per node

$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} \cancelto{0}{u_1} \\ \cancelto{0}{u_2} \\ u_3 \\ u_4 \\ \cancelto{0}{u_5} \\ \cancelto{0}{u_6} \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ \cancelto{0}{Q_3^1 + Q_1^2} \\ \cancelto{0}{Q_4^1 + Q_2^2} \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

homogenous

no point load acting

Condensed Eqn:

$$\frac{2EI}{h^3}\begin{bmatrix} 12 & 0 \\ 0 & 4h^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ h \end{Bmatrix}$$

$$\boxed{\begin{aligned} u_3 &= \frac{q_0 h^4}{48 EI} \\ u_4 &= \frac{q_0 h^3}{96 EI} \end{aligned}}$$

$$Q_1^1 = -\frac{q_0 h}{12} - (6u_3 + 3hu_4)\frac{2EI}{h^3} = -\frac{13}{16} q_0 h$$

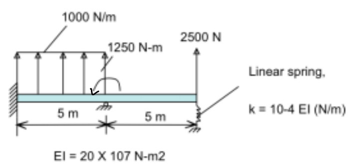
$$Q_2^1 = -\frac{q_0 h}{12} + (3hu_3 + h^2 u_4)\frac{2EI}{h^3} = \frac{11}{48} q_0 h^2$$

$$Q_3^2 = -\frac{2EI}{h^3}(-6u_3 + 3hu_4) = -\frac{3}{16} q_0 h$$

$$Q_4^2 = -\frac{2EI}{h^3}(-3hu_3 + h^2 u_4) = -\frac{5}{48} q_0 h^2$$

Example Problem 2 (# 5.15 Text)

Use the minimum number of Euler-Bernoulli beam finite elements to analyze the beam problem shown. Determine the unknown displacements and rotations.



2 element

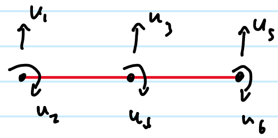
$$[k^1] = [k^2] = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

$$f^1 = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

$$f^2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assembled eqn: $\uparrow u_1 \quad \uparrow u_3 \quad \uparrow u_5$

Assembled eqn:



homogeneous

$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} \cancel{u_1} \rightarrow 0 \\ \cancel{u_2} \rightarrow 0 \\ \cancel{u_3} \rightarrow 0 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ \cancel{Q_4^1 + Q_2^2} \rightarrow -1250 \\ \cancel{Q_3^2} \rightarrow 2500-(10^{-4}EI)\,u_5 \\ \cancel{Q_4^2} \rightarrow 0 \end{Bmatrix}$$

u^k

Condensed Eqn:

$$\frac{2EI}{h^3}\begin{bmatrix} 4h^2 & 3h & h^2 \\ 3h & 6 & 3h \\ h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} h \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -1250 \\ 2500-(10^{-4}EI)\,u_5 \\ 0 \end{Bmatrix}$$

$$\frac{2EI}{h^3}\begin{bmatrix} 4h & 3h & h^2 \\ 3h & 6+\frac{h^3}{2}10^{-4} & 3h \\ h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} h \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -1250 \\ 2500 \\ 0 \end{Bmatrix}$$

$$\boxed{\begin{aligned} u_4 &= -.7237 \times 10^{-4} \text{ rad} \\ u_5 &= .087a \times 10^{-2} \text{ m} \\ u_6 &= -2275 \times 10^{-3} \text{ rad} \end{aligned}}$$

Homework 4

Wednesday, October 30, 2024 4:16 PM

C-Homework+4

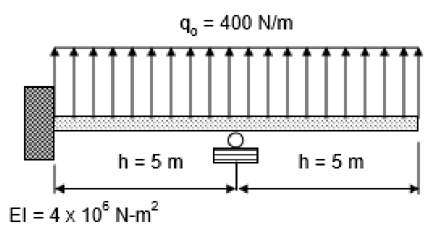
Due Date: November 5,2024 NAME: Easton Ingram

HOMEWORK SET # 4

ME/AE 5212 INTRODUCTION TO FINITE ELEMENT ANALYSIS

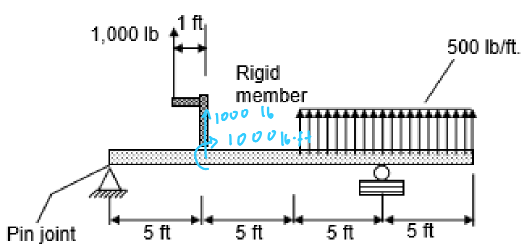
Solve the following beam problems using Euler-Bernoulli beam theory. Use the minimum number of elements. Determine the unknown displacements and slopes.

1.



(10 points)

2.



EI = constant

(10 points)

1. Const EI, Const q_0, $k_f = 0$

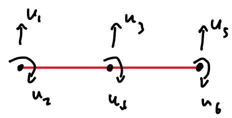
2 elements

$$K^1 = K^2 = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \end{bmatrix}$$

$$K^1 = K^2 = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

$$f^1 = f^2 = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

Assembled eqn:



$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} \cancel{u_1} \to 0 \\ \cancel{u_2} \to 0 \\ \cancel{u_3} \to 0 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6+6 \\ 0 \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ \cancel{Q_4^1 + Q_2^2} \to 0 \\ \cancel{Q_3^2} \to 0 \\ \cancel{Q_4^2} \to 0 \end{Bmatrix}$$

no point load

$$\frac{2EI}{h^3}\begin{bmatrix} 4h^2 & 3h & h^2 \\ 3h & 6 & 3h \\ h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 0 \\ 6 \\ h \end{Bmatrix}$$

Solving,

$$\boxed{\begin{aligned} u_4 &= -.001302 \\ u_5 &= .014323 \text{ m} \\ u_6 &= -.003385 \end{aligned}}$$

2. 4 elements

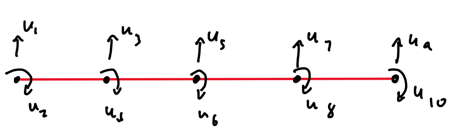
$$K^1 = K^2 = K^3 = K^4 = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

$$f^1 = f^2 = 0$$

$$f^3 = f^4 = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

$$\begin{matrix} 6 & -3h & -6 & -3h \\ 3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{matrix}$$

Assembled eqn:



$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 & 0 & 0 & 0 & 0 \\ 3h & 2h^2 & 3h & h^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h & 0 & 0 & 0 & 0 \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 3h & 6+6 & 3h-3h & -6 & -3h & 0 & 0 \\ 0 & 0 & -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ 0 & 0 & 0 & 0 & -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & 0 & 0 & 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} \cancel{u_1} \to 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \cancel{u_7} \to 0 \\ u_8 \\ u_9 \\ u_{10} \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ -h \\ 6+6 \\ h-h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ \cancel{Q_2^1} \to 0 \\ \cancel{Q_3^1 + Q_1^2} \to 1000 \text{ lb} \\ \cancel{Q_4^1 + Q_2^2} \to 1000 \text{ lb·ft} \\ \cancel{Q_3^2 + Q_1^3} \to 0 \\ \cancel{Q_4^2 + Q_2^3} \to 0 \\ Q_3^3 + Q_1^4 \\ \cancel{Q_4^3 + Q_2^4} \to 0 \\ \cancel{Q_3^4} \to 0 \\ \cancel{Q_4^4} \to 0 \end{Bmatrix}$$

Condensed Eqn:

$$\begin{bmatrix} \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 3h & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix} \begin{Bmatrix} \vdots \\ u_9 \\ u_{10} \end{Bmatrix} \quad \begin{Bmatrix} \vdots \\ 6 \\ h \end{Bmatrix} \quad \begin{Bmatrix} Q_4 + Q_2 = 0 \\ Q_3^d = 0 \\ Q_4^d = 0 \end{Bmatrix}$$

Condensed Eqn:

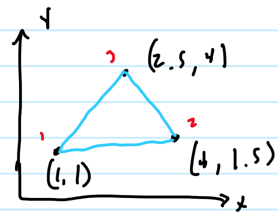
$$\frac{2EI}{l^3}\begin{bmatrix} 2h^2 & 3h & h^2 & 0 & 0 & 0 & 0 & 0 \\ 3h & 12 & 0 & -6 & -3h & 0 & 0 & 0 \\ h^2 & 0 & 4h^2 & 3h & h^2 & 0 & 0 & 0 \\ 0 & -6 & 3h & 12 & 0 & -3h & 0 & 0 \\ 0 & -3h & h^2 & 0 & 4h^2 & h^2 & 0 & 0 \\ 0 & 0 & 0 & -3h & h^2 & 4h^2 & 3h & h^2 \\ 0 & 0 & 0 & 0 & 0 & 3h & 6 & 3h \\ 0 & 0 & 0 & 0 & 0 & h^2 & 3h & 2h^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \\ u_9 \\ u_{10} \end{Bmatrix} = \frac{q_0 l}{12}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ -h \\ 0 \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving,

$$\boxed{\begin{aligned} u_2 &= -442.55/EI \\ u_3 &= 6556.83/EI \\ u_4 &= 491.38/EI \\ u_5 &= 1707797/EI \\ u_6 &= -4527/EI \\ u_8 &= -5926.64/EI \\ u_9 &= 68945.712/EI \\ u_{10} &= -16793.31/EI \end{aligned}}$$

2D

Tuesday, November 5, 2024 11:51 AM



$$\psi_i^e = \frac{1}{2A_e}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

$$\alpha_i = x_j y_k - x_k y_j$$

$$\beta_i = y_j - y_k$$

$$\gamma_i = x_k - x_j$$

$$2A_e = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1.5 \\ 1 & 2.5 & 4 \end{vmatrix} = 8.25 \quad 7.$$

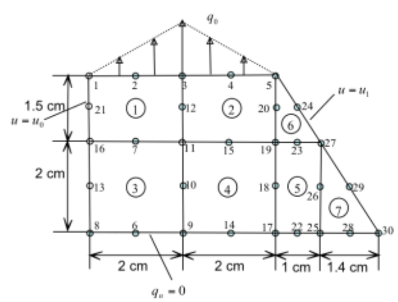
$$\alpha_1 = x_2 y_3 - x_3 y_2 = 12.25$$

$$\alpha_2 = x_3 y_1 - x$$

Ex.

Example Problem 2 (# 9.14 Text)

Compute the global force vector corresponding to the non-zero specified boundary flux for the finite element mesh of quadratic elements shown in figure.



$$Q_i^e = \oint_{\Gamma_e} q_n \psi_i \, ds$$

$$\text{Ch3}\ \begin{cases} \psi_1 = \left(1 - \frac{2\bar{x}}{h}\right)\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_2 = 4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_3 = -\frac{\bar{x}}{h}\left(1 - 2\frac{\bar{x}}{h}\right) \end{cases}$$

$$q_n = q_0 \frac{\bar{x}}{h}$$

$$Q_1^1 = \int_0^h q_n \psi_1 \, ds = \int_0^h \left(q_0 \frac{\bar{x}}{h}\right)\left(1 - \frac{2\bar{x}}{h}\right)\left(1 - \frac{\bar{x}}{h}\right) d\bar{x}$$

$$\left(1 + \frac{2\bar{x}^2}{h^2} - \frac{3\bar{x}}{h}\right) q_0 \frac{\bar{x}}{h}$$

$$q_0 \frac{\bar{x}}{h} + q_0 \frac{2\bar{x}^3}{h^3} - \frac{3 q_0 \bar{x}^2}{h^2}$$

$$\frac{1}{2} q_0 \frac{\bar{x}^2}{h} + \frac{1}{2} q_0 \frac{\bar{x}^4}{h^3} - \frac{q_0 \bar{x}^3}{h^2} \Big|^h$$

$$\frac{1}{2} q_0 h + \frac{1}{2} q_0 h - q_0 h$$

$$= 0$$

$$Q_2^1 = \int_0^h q_n \psi_2 \, ds = \int_0^h q_0 \frac{\bar{x}}{h} \frac{4\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) d\bar{x} = \frac{q_0 h}{3}$$

$$Q_3^1 = \int_0^h q_n \psi_3 \, ds = \int_0^h q_0 \frac{\bar{x}}{h}\left[-\frac{\bar{x}}{h}\left(1 - \frac{2\bar{x}}{h}\right)\right] dx = \frac{q_0 h}{6}$$

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_0 h/3 \\ q_0 h/6 + q_0 h/6 \\ q_0 h/3 \\ 0 \end{Bmatrix}$$

Symmetry for element 2

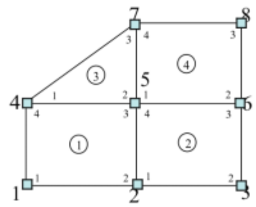
$$Q_3^2 = \frac{q_0 h}{6}$$

$$Q_4^2 = \frac{q_0 h}{3}$$

$$Q_5^2 = 0$$

Example Problem 3

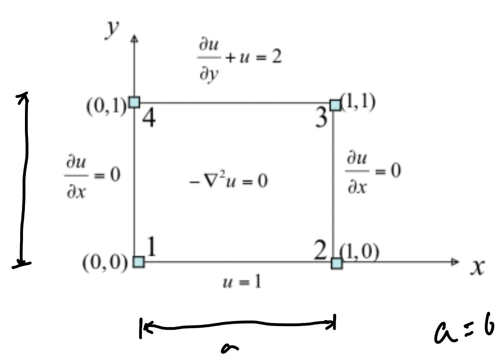
Give the assembled stiffness matrix and force vector for the finite element mesh shown. Assume one degree of freedom per node. The answer should be in terms of element matrix K_{ij}^e coefficients.



$$
K = \begin{array}{c c}
\begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} & \\
\begin{bmatrix}
K^1_{11} & K^1_{12} & 0 & K^1_{14} & K^1_{13} & 0 & 0 & 0 \\
 & K^2_{22}+K^2_{11} & K^2_{12} & K^2_{24} & K^1_{23}+K^2_{14} & K^2_{13} & 0 & 0 \\
 & & K^2_{22} & 0 & K^2_{24} & K^2_{23} & 0 & 0 \\
 & & & K^1_{44}+K^3_{11} & K^1_{24}+K^3_{12} & 0 & K^3_{13} & 0 \\
 & & & & K^1_{33}+K^2_{44} & K^2_{34}+K^4_{12} & K^3_{23}+K^4_{14} & 0 \\
 & & & & & K^2_{33}+K^4_{22} & K_{24} & K^4_{23} \\
 & & & & & & K^3_{33}+K^4_{44} & K^4_{34} \\
 & & & & & & & K^4_{33}
\end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{array}
\end{array}
$$

Example Problem 4 (# 9.21 Text)

Solve the Laplace equation for the unit square domain and boundary conditions given in Figure. Use one rectangular element.



$a = b = 1$

Modeling Eqn:

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u - f = 0$$

Laplace: $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = a_{00} = 0$, $f = 0$

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\nabla^2 u = 0$$

$$[K]\{u\} = \{\cancel{f}\}^{0} + \{Q\}$$

$$K_{ij}^e = \int_{\Omega_e}\left(\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y}\right)$$

$$= [S^{11}] + [S^{22}]$$

$$S^{11} = \frac{b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

$$+$$

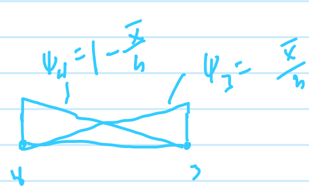
$$S^{22} = \frac{a}{6b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

$$=$$

$$K = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$

Assembled eqns:

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

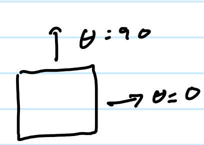


$$Q_i^e = \oint_\Gamma q_n \psi_i \, ds$$

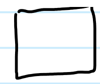
$$q_n = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$$

$$Q_3 = \int_0^1 \frac{\partial u}{\partial y} \psi_3 \, dx$$

$$n_x = \cos\theta$$
$$n_y = \sin\theta$$



$Q_3 = \int_0 \frac{\partial u}{\partial y} \varphi_3 \, dx$ $\quad n_x = \cos\theta$, $n_y = \sin\theta$



$\rightarrow \theta = 0$

$= \int_0^1 (2-u)\varphi_3 \, dx$ $\quad \sin 90 = 1, \cos 90 = 0$

$q_n = \frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial y} + u = 2$

$\frac{\partial u}{\partial y} = 2 - u$

$u = \sum_{j=1}^{n} u_j^e \varphi_j^e$

$u = u_3^1 \varphi_3 + u_4^1 \varphi_4$ $\quad x = \bar{x} \rightarrow \varphi_3 = x, \ \varphi_4 = 1 - x, \ h = 1$

$= u_3(x) + u_4(1-x)$

$Q_3 = \int_0^1 [2 - (x u_3 + (1-x) u_4)] \, x \, dx$

$= \frac{2}{2} - \frac{1}{3} u_3 - \frac{1}{6} u_4$

$1 - \frac{u_3}{3} - \frac{u_4}{6}$

$Q_4 = \int_0^1 (2-u)\varphi_4 \, dx$

$= \int_0^1 [2 - (x u_3 + (1-x) u_4)] (1-x) \, dx$

$= 1 - \frac{u_3}{6} - \frac{u_4}{3}$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ 1 - \frac{u_3}{3} - \frac{u_4}{6} \\ 1 - \frac{u_3}{6} - \frac{u_4}{3} \end{Bmatrix}$$

Solving,

$$\boxed{\begin{aligned} u_3 &= 1.5 \\ u_4 &= 1.5 \\ Q_1 = Q_2 &= -.25 \end{aligned}}$$

Homework 5

Tuesday, November 12, 2024 11:14 AM

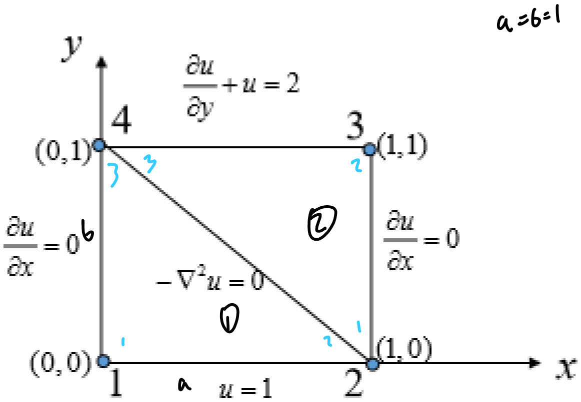
C-Homewor
k+5

Due: November 21,2024 NAME: Easton Ingram

HOMEWORK SET # 5
ME/AE 5212 Introduction to Finite Element Analysis

Solve the Laplace equation for the unit square domain and boundary conditions given in Figure. Use two triangular elements to solve the problem. Use the mesh obtained by joining points (1,0) and (0,1).

$a=b=1$



(20 points)

$$K^1 = \frac{1}{2ab}\begin{bmatrix} a^2+b^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$K^2 = \frac{1}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$K^2 = \frac{1}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

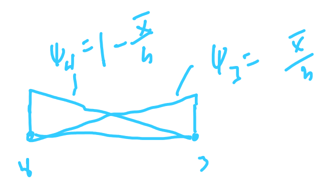
$$K = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}$$

$$K = \begin{bmatrix} K^1_{11} & K^1_{12} & 0 & K^1_{13} \\ K^1_{12} & K^1_{22}+K^2_{11} & K^2_{12} & K^1_{23}+K^2_{13} \\ 0 & K^2_{12} & K^2_{22} & K^2_{23} \\ K^1_{13} & K^1_{23}+K^2_{13} & K^2_{23} & K^1_{33}+K^2_{33} \end{bmatrix}\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

$$K = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1+1 & -1 & 0+0 \\ 0 & -1 & 2 & -1 \\ -1 & 0+0 & -1 & 1+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Assembled Eqn

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$



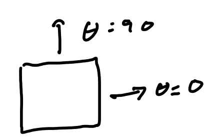
Same as class example

$$Q_i^e = \oint_\Gamma q_n \psi_i \, ds$$

$$q_n = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$$

$n_x = \cos\theta$

$n_y = \sin\theta$



$\sin 90 = 1$, $\cos 90 = 0$

$$q_n = \frac{\partial u}{\partial y}$$

$$Q_3 = \int_0^1 \frac{\partial u}{\partial y} \psi_3 \, dx$$

$$= \int_0^1 (2-u)\psi_3 \, dx$$

$$\frac{\partial u}{\partial y} + u = 2$$

$$\frac{\partial u}{\partial y} = 2 - u$$

$$u = \sum_{j=1}^{n} u_j^e \psi_j^e$$

$$u = u^1_3 \psi_3 + u^1_4 \psi_4$$
$$= u_3(x) + u_4(1-x)$$

$x = \bar{x} \rightarrow \psi_3 = x$, $\psi_4 = 1-x$, $h = 1$

$$Q_3 = \int_0^1 \left[2 - \left(x u_3 + (1-x) u_4\right)\right] x \, dx$$

$$= \frac{2}{2} - \frac{1}{3} u_3 - \frac{1}{6} u_4$$

$$1 - \frac{u_3}{3} - \frac{u_4}{6}$$

$$Q_4 = \int_0^1 (2-u)\,\psi_4 \, dx$$

$$= \int_0^1 \left[2 - \left(x u_3 + (1-x) u_4\right)\right] (1-x) \, dx$$

$$= 1 - \frac{u_3}{6} - \frac{u_4}{3}$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ 1 - \frac{u_3}{3} - \frac{u_4}{6} \\ 1 - \frac{u_3}{6} - \frac{u_4}{3} \end{Bmatrix}$$

Solving,

$$\boxed{\begin{array}{l} u_3 = 1.5 \\ u_4 = 1.5 \\ Q_1 = Q_2 = -.25 \end{array}}$$

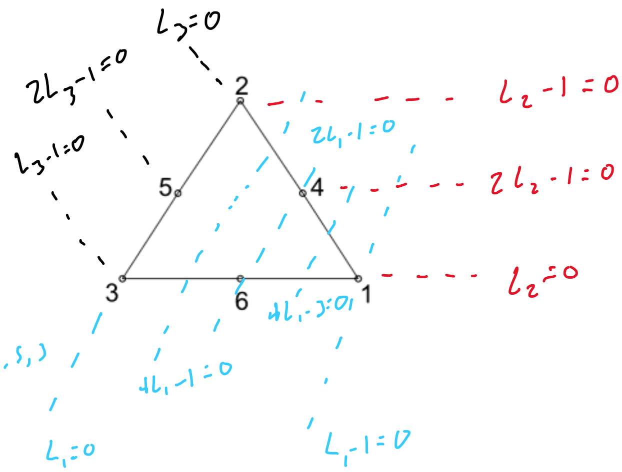
Higher Order Interpolation Functions Numerical Integration

Tuesday, November 12, 2024 11:45 AM

Example Problem
1
For a six node triangle, show that

$$\psi_1 = L_1(2L_1 - 1)$$

and $\psi_4 = 4L_1L_2$



ψ_1 vanishes at nodes 2, 5, 3

Assume
$\psi_1 = c\,L_1(2L_1 - 1)$

ψ_1 vanishes at nodes 4, 6

at node 1, $L_1 = 1$, $\psi_1 = 1$

$1 = c(1)(2\cdot 1 - 1)$

$1 = c$

$\boxed{\psi_1 = L_1(2L_1 - 1)}$

$\psi_4 = c L_1 L_2$

$L_2 = \frac{1}{2}$, $L_1 = \frac{1}{2}$, $\psi_4 = 1$

$c = 4$

$\boxed{\psi_4 = 4L_1L_2}$

$L_3 = 0$, $L_2 = \frac{1}{2}$, $\psi_4 = 1$

$\psi_4 = c(2L_3 - 1)(L_2)(L_2 - 1)$

$\psi_4 = c(-1)(\frac{1}{2})(-\frac{1}{2})$

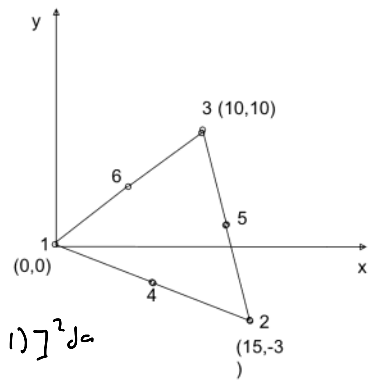
$c = 4$

Example Problem 2

Evaluate the area integrals

$$\iint_{Area} \psi_1^2 dA \quad \text{and} \quad \iint_{Area} \psi_1 \psi_4 dA$$

for the triangular element shown.



$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= 180$$

$$\iint_A \psi_1^2 \, dA = \iint_A [L_1(2L_1 - 1)]^2 da$$

$$= 4L_1^4 + L_1^2 - 4L_1^3) \, dA$$

$$\iint_A L_1^4 \, dA = \frac{4!\,0!\,0!}{(4+0+0+2)!} 2A = \frac{4(3)2}{6\cdot5\cdot4\cdot3\cdot2}(180) = \frac{1}{30}(180)$$

$$\iint_A L_1^2 \, dA = \frac{2!\,0!\,0!}{(2+0+0+2)!} 2A = \frac{2}{4\cdot3\cdot2}(180) = \frac{1}{12}(180)$$

$$\iint L_1^3 \, dA = \frac{3!\,0!\,0!}{(3+0+0+2)!} 2A = \frac{3(2)}{5(4)(3)(2)}(180) = \frac{1}{20}(180)$$

$$\iint \psi_1^2 \, dA = 6 + 15 - 9 = \left(\frac{4}{30} + \frac{1}{12} - \frac{4}{20}\right) 180 = 3$$

$$\iint \psi_1 \psi_4 \, dA = [L_1 (2L_1 - 1)(4L_1 L_2) \, dA$$

$$= \iint (8L_1^3 L_2 - 4L_1^2 L_2) \, dA$$

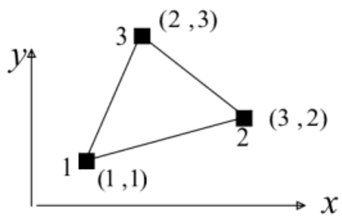
$$= \left(\frac{8}{120} - \frac{4}{60}\right) 180$$

Example Problem 3

Evaluate $\iint_{Area} xdA$ over an arbitrary triangle shown

(a) Using area integral

(b) One and four point Gauss quadrature formula



a. $\iint_A x\,da$

$x = L_1 x_1 + L_2 x_2 + L_3 x_3$

$\iint_A x\,dA = \iint_A (L_1 + 3L_2 + 2L_3)\,dA$

$\iint_A L_1^m L_2^n L_3^p\,dA = \frac{m!\,n!\,p!}{(m+n+p+2)!}\,2A$

$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 3$

$\iint x\,dA = \iint (L_1 + 3L_2 + 2L_3)\,dA$

$= \frac{2A}{3!} + 3\frac{2A}{3!} + 2\frac{2A}{3!}$

$= \frac{2A}{3!}(1+3+2)$

$\iint x\,dA = 3$

b. One point Gauss quadrature

$I = \iint_A F\,dA$

$= A\sum_{i=1}^{N} w_I F_I (L_{1I}, L_{2I}, L_{3I})$

$x = L_1 + 3L_2 + 2L_3$

$\iint x\,dA = A[w F_1]$

$F = L_1 + 3L_2 + 2L_3$

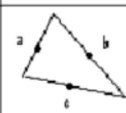
$\iint_A F\,dA = A[w F_1]$

$= A[1(\frac{1}{3} + \frac{3}{3} + \frac{2}{3})] = 3$

Numerical integration for triangular element

one point

four point

Order	Figure	Error	Points	Triangular coordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
Quadratic		$R = O(h^3)$	a b c	$\frac{1}{2}, \frac{1}{2}, 0$ $0, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$
Cubic		$R = O(h^4)$	a b c d	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $0.6, 0.2, 0.2$ $0.2, 0.6, 0.2$ $0.2, 0.2, 0.6$	$-\frac{27}{48}$ $\frac{25}{48}$ (b, c, d)
Quintic		$R = O(h^6)$	a b c d e f g	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $\alpha_1, \beta_1, \beta_1$ $\beta_1, \alpha_1, \beta_1$ $\beta_1, \beta_1, \alpha_1$ $\alpha_2, \beta_2, \beta_2$ $\beta_2, \alpha_2, \beta_2$ $\beta_2, \beta_2, \alpha_2$ with $\alpha_1 = 0.0597158717$ $\beta_1 = 0.4701420641$ $\alpha_2 = 0.7974269853$ $\beta_2 = 0.1012865073$	0.2250000000 0.1323941527 (b, c, d) 0.1259391805 (e, f, g)

Four point:

$\iint_A x\,dA = A[w_1 F_1 + w_2 F_2 + w_3 F_3 + w_4 F_4]$

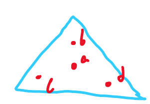
$x = L_1 + 3L_2 + 2L_3$

$-\frac{27A}{48}(\frac{1}{3} + \frac{3}{3} + \frac{2}{3}) + \frac{25}{48}A[(1(.6) + 3(0.2) + 2(0.2))$

$+ (1(.2) + 3(.6) + 2(.2))$

$+ (1(.2) + 3(.2) + 2(.6))]$

$= 3$



a: $L_1 = L_2 = L_3 = \frac{1}{3}$

b: $L_1 = .6$, $L_2 = .2$, $L_3 = .2$

c: $L_1 = .2$, $L_2 = .6$, $L_3 = .2$

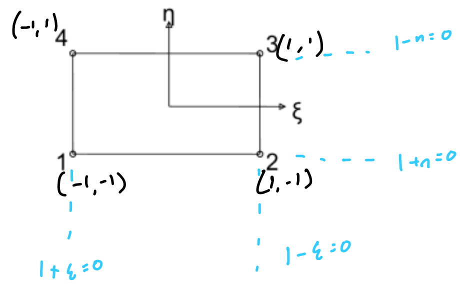
d: $L_1 = .2$, $L_2 = .2$, $L_3 = .6$

$w_1 = \frac{-27}{48}$, $w_2 = w_3 = w_4 = \frac{25}{48}$

Example Problem 4

For a four noded rectangular element, show that

$$\hat{\psi}_1 = \frac{1}{4}(1-\xi)(1-\eta)$$



$\psi_1 \quad (x_i, y_i) = \quad \delta_{ij}$

$\sum_{i=1} \psi_i = 1$

$\delta_{ij} = 1$ if $i=j$

$\delta_{ij} = 0$ if $i \neq j$

natural coordinates → $\hat{\psi}_1$ must vanish along sides $\xi=1$ and $\eta=1$

$\psi_1 = c(1-\xi)(1-\eta)$ ← vanishes at nodes 3 and 4

vanishes at nodes 2 and 3

$1 = c(1--1)(1--1)$

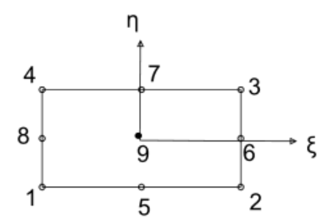
$c = \frac{1}{4}$

$$\boxed{\psi_1 = \frac{1}{4}(1-\xi)(1-\eta)}$$

Example Problem 5

For a nine noded rectangular element, show that

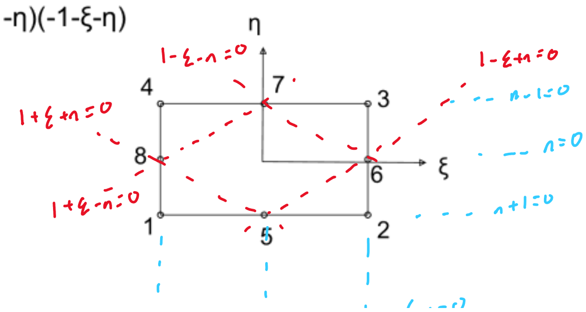
$$\hat{\psi}_1 = \frac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$$



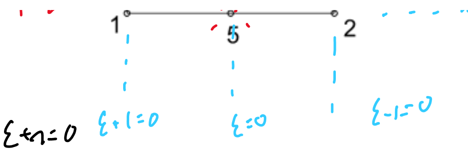
Example Problem 6

For an eight noded rectangular element show that

$$\hat{\psi}_1 = \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta)$$



ψ_1 vanishes along lines



ψ_1 vanishes along lines

$1-\xi=0$, $1-\eta=0$, $1+\xi+\eta=0$

$$\psi_1 = c\,(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$1 = c\,(1--1)(1--1)(1-1-1)$$

$$c = -\frac{1}{4}$$

$$\psi = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

A-Final+Exam-FEA

Thursday, December 5, 2024 10:56 AM

A-Final+Exam-FEA

NAME: Easton Ingram

ME/AE 5212
INTRODUCTION TO FINITE ELEMENT ANALYSIS
FINAL EXAM

Guidelines:

1. Exam is open book (Class notes, HW and textbook)
2. Need handwritten solution with all the steps and only calculator is allowed.
3. The test must be completed individually **in one sitting.**
4. Exam duration is 1 hr 30 min (11 am – 12:30 pm).
5. Submit the solution along with this cover page (with full name and signature) as a **single pdf file** in canvas.

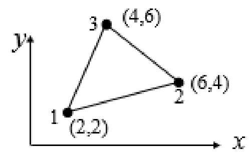
Missouri S&T Student Academic Conduct and Honor Code Statement:

I affirm that I have not given or received any unauthorized help on this exam, and that this is my own work.

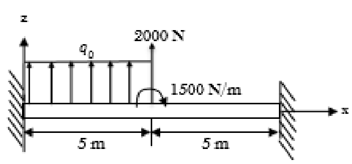
SIGNATURE Easton Ingram

1. Evaluate $\iint xy\, dA$ over an arbitrary triangle shown using **one-point** and **three-point** Gauss quadrature

(30 Points)



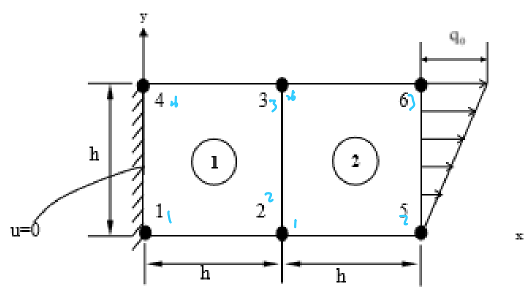
2. For the beam shown, determine the **primary unknowns** using Euler-Bernoulli beam element. Use **two elements** to model the full beam. **(30 points)**



E = 200 GPa
$I = 4 \times 10^{-4}\ m^4$
$q_0 = 150$ N/m

3. For the Laplace equation, $-\nabla^2 u = 0$, on a rectangular domain shown in figure, give the **global condensed equation**. Need not solve for unknowns.

(25 points)



4. TRUE OR FALSE **(15 Points)**

(a) For axisymmetric problem, the stress σ_θ is zero

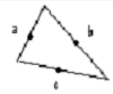
(b) Serendipity element is a nine-node rectangular element

(c) 1D domain can be represented using only one geometric shape

(d) Solution of Laplace's equation is a single variable problem

(e) Positions of the sampling points and the weights are not optimized in Newton-Cotes quadrature

Numerical integration for triangular element

Order	Figure	Error	Points	Triangular coordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
Quadratic		$R = O(h^3)$	a	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{1}{3}$
			b	$0, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}$
			c	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$
Cubic		$R = O(h^4)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$-\frac{27}{48}$
			b	0.6, 0.2, 0.2	$\frac{25}{48}$
			c	0.2, 0.6, 0.2	
			d	0.2, 0.2, 0.6	
Quintic		$R = O(h^6)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	0.2250000000
			b	$\alpha_1, \beta_1, \beta_1$	0.1323941527
			c	$\beta_1, \alpha_1, \beta_1$	
			d	$\beta_1, \beta_1, \alpha_1$	
			e	$\alpha_2, \beta_2, \beta_2$	0.1259391805
			f	$\beta_2, \alpha_2, \beta_2$	
			g	$\beta_2, \beta_2, \alpha_2$	
				with $\alpha_1 = 0.0597158717$, $\beta_1 = 0.4701420641$, $\alpha_2 = 0.7974269853$, $\beta_2 = 0.1012865073$	

1. $2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 6 & 4 \\ 1 & 4 & 6 \end{vmatrix} = 12$

$A = 6$

$\iint xy \, dA$

one point:

$I = \iint_A F \, dA$

$= A \sum_{i=1}^{N} w_I F_I (L_{1I}, L_{2I}, L_{3I})$

$xy = L_1 x_1 y_1 + L_2 x_2 y_2 + L_3 x_3 y_3$

$= 4L_1 + 24L_1 + 24L_3$

$\iint xy \, dA = A \; w_1 F_1$

$= 6 \left[1 \left(\frac{4}{3} + \frac{24}{3} + \frac{24}{3}\right)\right]$

$\boxed{= 104}$

three point:

$\iint_A F \, dA = A[w_1 F_1 + w_2 F_2 + w_3 F_3]$

$= \frac{1}{3} A \Big[\left[4\left(\frac{1}{2}\right) + 24\left(\frac{1}{2}\right) + 0\right] + \left[4(0) + 24\left(\frac{1}{2}\right) + 24\left(\frac{1}{2}\right)\right]$
$+ \left[4\left(\frac{1}{2}\right) + 24(0) + 24\left(\frac{1}{2}\right)\right]\Big]$

$= \frac{1}{3}(6)[52]$

$= \boxed{104}$

2. Modeling Eqn:

$[K^e]\{\Delta^e\} = \{F^e\}$

$EI = \text{const}, \; h = \text{const}, \; k_f = 0$

$$[K^e] = \frac{2E_eI_e}{h_e^3} \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

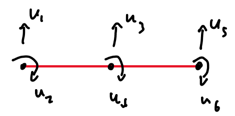
$$k_1 = k_2 = \frac{2EI}{h^3} \begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3 \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

$$F^e = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$F_1 = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

$$F_2 = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

Assembled Eqn:



2 DOF per node

homogeneous

$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} u_1 \to 0 \\ u_2 \to 0 \\ u_3 \\ u_4 \\ u_5 \to 0 \\ u_6 \to 0 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \to 2000\,N \\ Q_4^1 + Q_2^2 \to 1500\,N\,m \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

Condensed Eqn:

$$\frac{2EI}{h^3}\begin{bmatrix} 12 & 0 \\ 0 & 4h^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{q_0 h}{12}\begin{Bmatrix} 6 \\ h \end{Bmatrix} + \begin{Bmatrix} 2000 \\ 1500 \end{Bmatrix}$$

$$\frac{2(200)(4\times10^4)}{125}\begin{bmatrix} 12 & 0 \\ 0 & 100 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{150(5)}{12}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} + \begin{Bmatrix} 2000 \\ 1500 \end{Bmatrix}$$

solving,

$$\boxed{\begin{aligned} u_3 &= .1546\ m \\ u_4 &= .01476\ rad \end{aligned}}$$

3.

$$K_{ij}^e = [S^{11}] + [S^{22}]$$

$$K^e = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$

$$K = \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 & 0 & 0 \\ & K_{22}^1 + K_{11}^2 & K_{23}^1 + K_{14}^2 & K_{24}^1 & K_{12}^2 & K_{13}^2 \\ & & K_{33}^1 + K_{44}^2 & K_{34}^1 & K_{42}^2 & K_{34}^2 \\ & & & K_{44}^1 & 0 & 0 \\ & & & & K_{22}^2 & K_{23}^2 \\ & & & & & K_{33}^2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}$$

$$K = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 \\ & 4+4 & -2+-1 & -2 & -1 & -2 \\ & & 4+4 & -1 & -2 & -1 \\ & & & 4 & 0 & 0 \\ & & & & 4 & -1 \end{bmatrix}$$

$$\frac{1}{6}\begin{bmatrix} & & 4+4 & -1 & -2 & -1 \\ & & & 4 & 0 & 0 \\ & & & & 4 & -1 \\ & & & & & 4 \end{bmatrix}$$

$$\varphi_1^e = 1 - \frac{x}{h}$$

$$\varphi_2^e = \frac{x}{h}$$

$$q_n = q_0 \frac{x}{h}$$

$$Q_2^2 = \int_0^h q_n \varphi_2 \, dx = \frac{1}{3}\frac{x^3}{h^2} q_0 \Big|_h = \frac{1}{3} q_0 h$$

$$Q_1^2 = \int_0^h q_n \varphi_1 \, dx = \frac{1}{2} q_0 \frac{x^2}{h} - \frac{1}{3} q_0 \frac{x^3}{h^2} \Big|_h = \frac{1}{2} q_0 h - \frac{1}{3} q_0 h = \frac{1}{6} q_0 h$$

Global eqn:

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 \\ -1 & 8 & -3 & -2 & -1 & -2 \\ -2 & -3 & 8 & -1 & -2 & -1 \\ -1 & -2 & -1 & 4 & 0 & 0 \\ 0 & -1 & -2 & 0 & 4 & -1 \\ 0 & -2 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_3^1 + Q_4^2 \\ Q_4^1 \\ Q_2^2 \\ Q_2^3 \end{Bmatrix}$$

Out of time..

4. a. True

b. False

c. True

d. True

e. True